(5) 1. Solve the system

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 + x_5 &= 1 \\ 2x_1 + x_2 + x_3 + 2x_4 - x_5 &= 2 \\ x_1 + 2x_2 - 4x_3 - 8x_4 + 5x_5 &= 1 \\ x_2 - 3x_3 - 6x_4 + 3x_5 &= 0 \end{aligned}$$

(5) 2. Let $A = \begin{bmatrix} 2 & 6 & -5 \\ -1 & -3 & 3 \\ 1 & 4 & -6 \end{bmatrix}$

(a) Find A^{-1}.

(b) Use your answer in part (a) to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 8 \\ -4 \\ 5 \end{bmatrix}$.

(6) 3. Let $\mathbf{u}_1 = \begin{bmatrix} x \\ x \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} x \\ 2 \\ x \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ x \\ -x \end{bmatrix}$

(a) For what value(s) of x will $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be linearly dependent?

(b) For what value(s) of x will $\{\mathbf{u}_1, \mathbf{u}_2\}$ be linearly dependent?

(c) For what value(s) of x is Span $\{\mathbf{u}_1, \mathbf{u}_2\}$ all of $\mathbb{R}^3$?

(d) For what value(s) of x is Span $\{\mathbf{u}_1, \mathbf{u}_2\}$ a line in $\mathbb{R}^3$?

(4) 4. For each of the following, find an example or explain why no such matrix is possible.

(a) A 2×3 matrix A so that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(b) A 2×3 matrix A where every entry is either 1 or -1 so that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is **not** onto.

(c) A matrix A such that A^2 is invertible but A is not.

(d) A 2×2 nonzero matrix A such that $A^2 = 0$.

(6) 5. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates points by $\pi/4$ radians counterclockwise. Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects the points across the line $y = -x$.

(a) Give the standard matrices of T_1 and T_2.

(b) Give the standard matrix of $T_1 \circ T_2$.

(c) Let $\mathcal{S}$ denote the unit square in $\mathbb{R}^2$, that is $\mathcal{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1 \right\}$
Draw pictures of $\mathcal{S}$ and $(T_1 \circ T_2)(\mathcal{S})$.

(5) 6. Let $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

(a) For what value(s) of k is $\begin{bmatrix} 3 \\ k \end{bmatrix}$ in Col A?

(b) For what value(s) of k is $\begin{bmatrix} 3 \\ k \end{bmatrix}$ in $\text{Nul}\, A$?

(c) Give a basis for $\text{Nul}\, A^2$.

(d) Is $\text{Nul}\, A = \text{Nul}\, A^2$? Justify your answer.

(3) 7. Suppose A and B are $n \times n$ matrices where A has linearly independent columns and B is invertible.

(a) Simplify $(BAB^{-1})^2$.

(b) Simplify $(BAB^{-1})^{-1}$.

(c) Does BAB^{-1} have linearly independent columns? Justify your answer.

(7) 8. Fill in the blanks. The missing word is **must**, **might** or **cannot**.

(a) If $\mathbf{y} \in \text{Col}\, A$ then $A\mathbf{x} = \mathbf{y}$ ________ be inconsistent.

(b) If $\mathbf{y} \in \text{Col}\, A$ then $\mathbf{y}$ ________ be in $\text{Nul}\, A$.

(c) If $\mathbf{y} \in \text{Col}\, A$ then $\mathbf{y}$ ________ be in $\text{Row}\, A^T$.

(d) If $\mathbf{x} \in \text{Col}\, A$ and $\mathbf{y} \in \text{Col}\, A$ then $\mathbf{x} + \mathbf{y}$ ________ be in $\text{Col}\, A$.

(e) Suppose A is a 5×7 matrix then $\text{Row}\, A$ and $\text{Col}\, A$ ________ have the same dimension.

(f) Suppose A is a 5×7 matrix then $\text{Nul}\, A$ ________ be 3 dimensional.

(g) Suppose A is a 5×7 matrix of rank 4, then $\text{Nul}\, A^T$ ________ be 3 dimensional.

(6) 9. Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 6 & -17 & 13 \end{bmatrix}$

(a) Find lower triangular matrix L and upper triangular matrix U so that $A = LU$.

(b) Do the same for A^T. (Hint: No additional computation is required.)

(c) What is $\det A$?

(d) Find an elementary matrix E such that $EA = \begin{bmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 0 & -8 & 1 \end{bmatrix}$.

(6) 10. Let U be an $n \times n$ matrix which is partitioned as $U = \begin{bmatrix} 0 & I \\ A & B \end{bmatrix}$.

(a) Assume A is invertible. Write U^{-1} as a partitioned matrix.

(b) Use part (a) to find the inverse of M where $M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 7 & 5 & 3 & 2 & 6 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}$

(7) 11. Let $A = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 4 & 3 & 5 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

(a) Find $\det A$.

(b) Use Cramer's Rule to solve $A\mathbf{x} = \mathbf{b}$ for x_4 only.

(c) What is $\det(-2A^{-1})$?

(d) What is $\det(A^{-1}A^TA)$?

(7) 12. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 2d \text{ and } bc \leq 0 \right\}$.

(a) Is $\mathbf{0}$ in V?

(b) Is V closed under scalar multiplication? Justify your answer. No credit is given without a justification.

(c) Is V closed under addition? Justify your answer. No credit is given without a justification.

(d) Is V a subspace of $\mathbb{R}^4$?

(4) 13. Let $V = \{p(x) \in \mathbb{P}_2 : p'(1) = p(1) \text{ and } p'(2) = p(2)\}$. Given that V is a subspace of $\mathbb{P}_2$ find a basis for V and state the dimension of V.

(6) 14. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a \\ -2 \\ b \end{bmatrix}$ find the following:

(a) cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$,

(b) the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$,

(c) the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

(d) all values of a and b such that $\mathbf{x}$ is orthogonal to $\mathbf{u}$.

(6) 15. Given the plane $\mathcal{P} : x + y - z = 11$ and the line $\mathcal{L} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ find the following:

(a) the point of intersection of $\mathcal{P}$ and $\mathcal{L}$,

(b) the distance from the point $Q(2, -1, 3)$ to the plane $\mathcal{P}$,

(c) the distance from the point $R(1, 0, 1)$ to the line $\mathcal{L}$.

(5) 16. (a) Show that the lines

$$\mathcal{L}_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + s \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \qquad \mathcal{L}_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

intersect in a point and find the point of intersection.

(b) Find a standard equation of the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$.

(4) 17. (a) Show that if $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\|$ then $\mathbf{x} \cdot \mathbf{y} \leq 0$

(b) Prove the identity $\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 4\mathbf{x} \cdot \mathbf{y}$

(4) 18. Suppose $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$ is an $n \times n$ matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that each column of A is a unit vector. (Hint: consider $\mathbf{a}_i = A\mathbf{e}_i$.)

(b) Show that any two different columns of A, $\mathbf{a}_i$ and $\mathbf{a}_j$, are orthogonal. (Hint: Consider the result in part (a) and $\|\mathbf{a}_i + \mathbf{a}_j\|^2$.)

(c) Show that $A^T A = I_n$.

(4) 19. A matrix X is called a **weak generalized inverse** of A if

$$AXA = A$$

(a) For what value of k is $\begin{bmatrix} k & k \\ k & k \\ k & k \end{bmatrix}$ a weak generalized inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$?

For parts (b) and (c), suppose that X is a weak generalized inverse of $m \times n$ matrix A (so you know that $AXA = A$ even though A is not necessarily invertible).

(b) Show that if $\mathbf{y}$ is any vector in $\mathbb{R}^n$, then $(I - XA)\mathbf{y}$ is in $\operatorname{Nul} A$.

(c) Show that if the system $A\mathbf{x} = \mathbf{b}$ is consistent then $X\mathbf{b}$ will be a solution to this system.