Answers to Math NYC-Final Exam (May 2011)

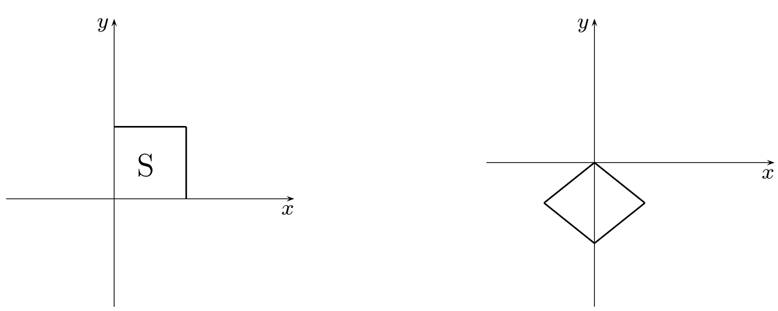
1. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ where $s, t \in \mathbb{R}$

2. $A^{-1} = \begin{bmatrix} -6 & -16 & -3 \\ 3 & 7 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3. (a) When $\begin{vmatrix} x & x & 1 \\ x & 2 & x \\ 2 & x & -x \end{vmatrix} = x^2 - 4 = 0$ so for $x = \pm 2$
   (b) $x = 2$
   (c) Never since two vectors can not generate $\mathbb{R}^3$.
   (d) $x = 2$

4. (a) Not possible since A can not have more than two linearly independent columns.
   (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$ among others
   (c) Not possible since $\det A^2 \neq 0$ implies that $\det A \neq 0$ implying that A is invertible as well.
   (d) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ among others

5. (a) $A_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
   (b) $A_1 A_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$
   (c) Let $x = x_1$ and $y = x_2$. The regions are then as follows:



6. (a) $k = 6$
   (b) $k = \frac{3}{2}$
   (c) $\mathfrak{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$
   (d) Yes, $A\mathbf{x} = 0 \to A^2\mathbf{x} = 0$ which implies that $\operatorname{Nul} A \subseteq \operatorname{Nul} A^2$. On the other hand, $\operatorname{rank} A = \operatorname{rank} A^2 = 1$ implying that $\operatorname{Nul} A$ and $\operatorname{Nul} A^2$ have the same dimension, so $\operatorname{Nul} A = \operatorname{Nul} A^2$.

7. (a) $(BAB^{-1})^2 = BAB^{-1}BAB^{-1} = BA^2B^{-1}$
   (b) $(BAB^{-1})^{-1} = BA^{-1}B^{-1}$
   (c) Yes since $\det(BAB^{-1}) = \det B \det A \det B^{-1} = \det A \neq 0$.

8. (a) **cannot**
   (b) **might**
   (c) **must**
   (d) **must**
   (e) **must**
   (f) **might**
   (g) **cannot**

9. (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

   (b) $A^T = U^T L^T = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & 0 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

   (c) $\det A = \det L \det U = 1(2)(4)(5) = 40$

   (d) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

10. (a) $U^{-1} = \begin{bmatrix} -A^{-1}B & A^{-1} \\ I & 0 \end{bmatrix}$

    (b) $M^{-1} = \begin{bmatrix} 1 & -1 & 7 & 3 & -5 \\ -2 & 1 & -11 & -4 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

11. (a) $\det A = 18$
    (b) $x_4 = \frac{\det A_4}{\det A} = \frac{-42}{18} = \frac{-7}{3}$
    (c) $\det(-2A^{-1}) = (-2)^4 \det A^{-1} = \frac{8}{9}$
    (d) $\det(A^{-1}A^TA) = \det A^{-1} \det A^T \det A = \det A = 18$

12. (a) Yes; $0 = 2(0)$ and $0(0) \leq 0$.

    (b) Yes; if $\mathbf{u} \in V$ then $\mathbf{u} = \begin{bmatrix} 2d \\ b \\ c \\ d \end{bmatrix}$ with $bc \leq 0$ and $k\mathbf{u} = \begin{bmatrix} 2kd \\ kb \\ kc \\ kd \end{bmatrix}$ with $k^2bc \leq 0$ so $k\mathbf{u} \in V$.

    (c) No, e.g., $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ -6 \\ 2 \end{bmatrix} \in V$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 8 \\ 1 \end{bmatrix} \in V$ yet $\mathbf{u} + \mathbf{w} = \begin{bmatrix} 6 \\ 2 \\ 2 \\ 3 \end{bmatrix}$ is not in V.

    (d) No, by part (c).

13. $\mathfrak{B} = \{x^2 - x + 1\}$ and $\dim V = 1$.

14. (a) $\cos\theta = \frac{18}{5\sqrt{14}}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}$, $A = \frac{\sqrt{26}}{2}$

(c) $V = 7$

(d) All a and b satisfying $3a + 2b = -2$

15. (a) $(3, 4, -4)$

(b) $D = \frac{13}{\sqrt{3}}$

(c) $d = \frac{\sqrt{5}}{3}$

16. (a) $(-17, -1, 1)$

(b) $x - 4y + 4z = -9$

17. (a)

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 \\ (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) &\leq \mathbf{x} \cdot \mathbf{x} \\ \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} &\leq \mathbf{x} \cdot \mathbf{x} \\ 2\mathbf{x} \cdot \mathbf{y} &\leq -\|\mathbf{y}\|^2 \\ \mathbf{x} \cdot \mathbf{y} &\leq 0 \end{aligned}$$

(b)

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) - (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} - (\mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}) \\ &= 4\mathbf{x} \cdot \mathbf{y} \end{aligned}$$

18. (a) $\|\mathbf{a}_i\| = \|A\mathbf{e}_i\| = \|\mathbf{e}_i\| = 1$

(b) $\|\mathbf{a}_i + \mathbf{a}_j\|^2 = \|\mathbf{a}_i\|^2 + 2\mathbf{a}_i \cdot \mathbf{a}_j + \|\mathbf{a}_j\|^2 = 2 + 2\mathbf{a}_i \cdot \mathbf{a}_j$

Also $\|\mathbf{a}_i + \mathbf{a}_j\|^2 = \|A(\mathbf{e}_i + \mathbf{e}_j)\|^2 = \|\mathbf{e}_i + \mathbf{e}_j\|^2 = 2$ for $i \neq j$

So $2 + 2\mathbf{a}_i \cdot \mathbf{a}_j = 2$ this implies that $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for $i \neq j$

(c) $A^T A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} [\mathbf{a}_1 \;\; \mathbf{a}_2 \cdots \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix} = I_n$

19. (a) $k \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$k \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so $k = \frac{1}{6}$

(b) $A(I - XA)\mathbf{y} = A\mathbf{y} - (AXA)\mathbf{y} = A\mathbf{y} - A\mathbf{y} = 0$

(c) $AX\mathbf{b} = AX(A\mathbf{x}) = (AXA)\mathbf{x} = A\mathbf{x} = \mathbf{b}$