1. (a) Solve the system:

$$x_1 + x_2 - x_3 - 2x_4 + x_5 = 1$$
$$2x_1 + x_2 + x_3 + 2x_4 - x_5 = 2$$
$$x_1 + 2x_2 - 4x_3 - 8x_4 + 5x_5 = 1$$
$$x_2 - 3x_3 - 6x_4 + 3x_5 = 0$$

(b) Write the zero vector in $\mathbb{R}^4$ as a nontrivial linear combination of the columns of A, where A is the coefficient matrix for the system of equations in part a) of this question.

2. Let $A = \begin{bmatrix} 2 & 6 & -5 \\ -1 & -3 & 3 \\ 1 & 4 & -6 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Use your answer in part (a) to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 2y \\ x-1 \end{bmatrix}$.

(a) Is T linear? Justify.

(b) Is T one-to-one? Justify.

(c) Is T onto? Justify.

(d) Sketch the line $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ then find its image under T.

4. Give an example of each of the following. If no such example is possible, explain why.

(a) A 2×3 matrix A such that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(b) A 2×3 matrix A where every entry is either 1 or -1 such that the tranformation $\mathbf{x} \mapsto A\mathbf{x}$ is NOT onto.

(c) A matrix A such that A^2 is invertible but A is not.

(d) A nonzero matrix A such that $A^2 = 0$.

5. Let A and B be $n \times n$ matrices where B is invertible and A has linearly independent columns.

(a) Simplify $(BAB^{-1})^2$.

(b) Simplify $(BAB^{-1})^{-1}$.

(c) Does BAB^{-1} have linearly independent columns? Justify your answer.

6. Let $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$.

(a) For which value(s) of k is $\begin{bmatrix} 3 \\ k \end{bmatrix}$ in Col(A)?

(b) For which value(s) of k is $\begin{bmatrix} 3 \\ k \end{bmatrix}$ in Nul(A)?

(c) Give a basis for Nul(A^2).

(d) Is Nul(A) =Nul(A^2)? Justify your answer.

7. Fill in each blank with the missing word. In each case, the missing word is either, *must*, *might* or *cannot.*

(a) If $\mathbf{y} \in \mathrm{Col}(A)$ then $A\mathbf{x} = \mathbf{y}$ ____________ be inconsistent.

(b) If $\mathbf{y} \in \mathrm{Col}(A)$ then $\mathbf{y}$ ____________ be in $\mathrm{Nul}(A)$.

(c) If $\mathbf{y} \in \mathrm{Col}(A)$ then $\mathbf{y}$ ____________ be in Row (A^T).

(d) If $\mathbf{y} \in \mathrm{Col}(A)$ and $\mathbf{x} \in \mathrm{Col}(A)$ then $\mathbf{x} + \mathbf{y}$ ____________ be in $\mathrm{Col}(A)$.

(e) If A is a 5×7 matrix then $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$ ____________ have the same dimension.

(f) If A is a 5×7 matrix then $\mathrm{Nul}(A)$ ____________ be three-dimensional.

(g) If A is a 5×7 matrix of rank 4, then $\mathrm{Nul}(A^T)$ ____________ be three-dimensional.

(h) If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent then $\mathrm{Proj}_{\mathbf{u}}\mathbf{v}$ and $\mathrm{Proj}_{\mathbf{v}}\mathbf{u}$ ____________ be equal.

8. Let W be an $n \times n$ matrix that is partitioned as $W = \begin{bmatrix} 0 & I \\ A & B \end{bmatrix}$, where the matrix A is known to be invertible.

(a) Write W^{-1} as a partitioned matrix.

(b) Use part (a) to find M^{-1} where $M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & 3 & 2 & 6 \\ -1 & 2 & 2 & 1 & 5 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 6 & -17 & 13 \end{bmatrix}$.

(a) Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

(b) Do the same for A^T. (Hint: No additional computation is required.)

(c) Find an elementary matrix E such that $EA = \begin{bmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 0 & -8 & 1 \end{bmatrix}$.

10. Let $A = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 4 & 3 & 5 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

(a) Find $\det(A)$.

(b) Use Cramer's Rule to solve $A\mathbf{x} = \mathbf{b}$ for x_4 ONLY.

(c) What is $\det(A^{-1}A^T)$?

(d) What is $\det(A \cdot \mathrm{adj}(A))$?

(e) Find the determinant of $B = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 6 & 0 & 0 & 3 \\ 4 & 3 & 5 & 1 \\ 3 & -6 & -6 & 0 \end{bmatrix}$, noting that B is obtained from A by performing exactly two elementary row operations.

11. Let $\mathbf{u}_1 = \begin{bmatrix} x \\ x \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} x \\ 2 \\ x \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ x \\ -x \end{bmatrix}$.

(a) For which value(s) of x will $\{\mathbf{u}_1, \mathbf{u}_2\}$ be linearly dependent?

(b) For which value(s) of x will $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ be all of $\mathbb{R}^3$?

(c) For which value(s) of x is $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ a line in $\mathbb{R}^3$?

(d) For which value(s) of x will $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be linearly dependent?

12. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = 2c,\ bd \le 0 \right\}$.

    (a) Is $\mathbf{0} \in V$?

    (b) Is V closed under scalar multiplication? Justify.

    (c) Is V closed under vector addition? Justify.

    (d) Is V a subspace of $\mathbb{R}^4$?

13. Let $\mathcal{P} : x - 4y + 2z = 3$ be a plane in $\mathbb{R}^3$.

    (a) Does $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ define the same plane $\mathcal{P}$?

    (b) Find the equation of a line perpendicular to $\mathcal{P}$ passing through $Q(3, 1, 1)$.

    (c) Find the distance from $\mathcal{P}$ to $Q(3, 1, 1)$.

    (d) Find the cosine of the angle between $\mathcal{P}$ and the line: $t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

    (e) Is $\mathcal{P}$ a subspace of $\mathbb{R}^3$? Justify.

14. Given the parallel lines $\mathcal{L}_1 : \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathcal{L}_2 : \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, find:

    (a) An equation for the plane containing both $\mathcal{L}_1$ and $\mathcal{L}_2$.

    (b) The distance between $\mathcal{L}_1$ and $\mathcal{L}_2$.

    (c) The point on $\mathcal{L}_1$ that is closest to the point $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ on $\mathcal{L}_2$.

15. Let $V = \{p(x) \in \mathbb{P}_2 : p'(1) = p(1) \text{ and } p'(2) = p(2)\}$. Given that V is a subspace of $\mathbb{P}_2$, find a basis for V and state the dimension of V.

16. Suppose that $T : V_1 \to V_2$ is a one-to-one linear transformation and suppose that H is a nonzero subspace of the vector space V_1. Then $T(H)$, the set of all images of vectors in H under T, is a subspace of V_2.

    (a) Define what it means for a set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ to be a basis for H.

    (b) Prove that dim(H)=dim($T(H)$).

17. Suppose A is an $n \times n$ matrix such that $||A\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n$. Note that if $\mathbf{a}_i$ is the ith column of A and $\mathbf{e}_i$ is the i^{th} column of the identity matrix $I_{n\times n}$, then $\mathbf{a}_i = A\mathbf{e}_i$.

    (a) Show that each column of A is a unit vector.

    (b) Show that $||\mathbf{a}_i + \mathbf{a}_j||^2 = ||\mathbf{a}_i||^2 + ||\mathbf{a}_j||^2$ for any two columns $\mathbf{a}_i, \mathbf{a}_j$ of A. What can you conclude about the vectors $\mathbf{a}_i$ and $\mathbf{a}_i$? (Hint: Pythagoras!)

    (c) Show that $A^T A = I_{n\times n}$.

    (d) Give an example of a 2×2 matrix A (other than the identity matrix) such that $||A\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n$.

Solutions 1. a) $x_1 = -2x_3 - 4x_4 + 1$, $x_2 = 3x_3 + 6x_4$, x_3 is free, x_4 is free, $x_5 = 0$ b) $\mathbf{0} = -6\mathbf{a}_1 + 9\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4$ where $\mathbf{a}_i$ is the i^{th} column of A. 2. a) $A^{-1} = \begin{bmatrix} -6 & -16 & -3 \\ 3 & 7 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ b) $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ 3.a) No, since $T(\mathbf{0}) \neq \mathbf{0}$ b) Yes; prove that if $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ then $\mathbf{v}_1 = \mathbf{v}_2$ c) No, since for example the zero vector is not in Range(T) d) The image of the line is $(3, 2, 1) + t(-1, 4, -1)$ 4. a) Impossible b) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$ c) Impossible d) $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ 5. a) BA^2B^{-1} b)$B(BA)^{-1}$ c) Yes 6. a) $k = 6$ b) $k = 3/2$ c) $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ d) They are equal. 7. a) cannot b) might c) must d) must e) must f) might g) cannot h) cannot 8. a) $W^{-1} = \begin{bmatrix} -A^{-1}B & A^{-1} \\ I & 0 \end{bmatrix}$ b) $M^{-1} = \begin{bmatrix} 12 & 7 & 27 & -2 & -3 \\ 5 & 3 & 11 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ 9.a) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ b) $A^T = (LU)^T = U^T L^T$ c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ 10. a) 18 b) $-7/3$ c) 1 d) $(18)^4$ e) -18 11. a) $x = 2$ b) Impossible c) $x = 2$ d) $x = \pm 2$ 12. a) Yes b) Yes c) No d) No 13. a) No b) $(3, 1, 1) + t(1, -4, 2)$ c) $\frac{2\sqrt{21}}{21}$ d) The angle is $\frac{\pi}{2} - \cos^{-1}(-2/\sqrt{21})$ e) No. It doesn't pass through the origin. 14. a) $(0, 1, -1) + s(2, 2, 3) + t(2, 1, -3)$ b) $\frac{\sqrt{34^2 + 31^2 + 33^2}}{14}$ c) $\frac{1}{14}(-6, 11, -5)$ 15. $\mathcal{B} = \{x^2 - x + 1\}$ and dim(V)=1 16. a) The vectors in $\mathcal{B}$ are linearly independent and span H. b) Show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$ is a basis for $T(H)$. 17. a) $||\mathbf{a}_i|| = ||A\mathbf{e}_i|| = ||\mathbf{e}_i|| = 1$ b) $||\mathbf{a}_1 + \mathbf{a}_j||^2 = ||A\mathbf{e}_i + A\mathbf{e}_j||^2 = ||A(\mathbf{e}_i + \mathbf{e}_j)||^2 = ||\mathbf{e}_i + \mathbf{e}_j||^2 = ||\mathbf{e}_i||^2 + ||\mathbf{e}_j||^2 = ||A\mathbf{e}_i||^2 + ||A\mathbf{e}_j||^2 = ||\mathbf{a}_i||^2 + ||\mathbf{a}_j||^2$ Conclusion: $\mathbf{a}_i$ and $\mathbf{a}_j$ are orthogonal. c) The entries along the diagonal are of the form $\mathbf{a}_i \cdot \mathbf{a}_i = ||\mathbf{a}_i||^2 = 1$ and the entries off the diagonal are of the form $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ where $i \neq j$ since $\mathbf{a}_i$ and $\mathbf{a}_j$ are orthogonal. d) Any rotation matrix would work.