1. Solve each of the following systems, or explain why the system has no solution.

a) $\begin{cases} x - 4y = 17 \\ 2x + 5y = -5 \\ -3x - 7y = 6 \end{cases}$ b) $\begin{cases} 2x - 4y - 2z + 8w = -4 \\ -3x + 4y - z - 2w = 2 \\ -x + 3y + 3z - 9w = 5 \end{cases}$

2. Given the system $\begin{cases} 2x + y + 3z = a \\ x + 5y = b \\ 5x + 7y + 6z = c \end{cases}$

Find a relation between *a*, *b* and *c* such that the system is:
a) consistent b) inconsistent.

3. a) Find an equation of the plane, in the form $Ax + By + Cz = D$,
that passes through the points $(-1, 2, 4)$, $(3, -4, 1)$, $(1, -2, -1)$.
b) Is the point $(0, 3, 8)$ on the plane found in part a)? Justify.

4. A potter is making cups, vases and plates. Each cup uses 3 lb. of clay and 2 oz. of glaze, each vase uses 7 lb. of clay and 5 oz. of glaze and each plate uses 4 lb. of clay and 2 oz. of glaze. She has 178 lb. of clay and 122 oz. of glaze on hand. How many whole cups, vases and plates can she make if she uses all available clay and glaze?
   a) Define all the necessary variables *x*, *y*, and *z*, and set up the system of equations required to solve the problem.
   b) Find the general (parametric) solution for the system.
   c) Find one particular solution which is realistic.
   d) Is it possible to have a solution with no vases made?

5. Given $A^{-1} = \begin{bmatrix} 2 & 5 \\ 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -2 & -1 \\ 5 & 0 \\ 0 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}$

Find the following or explain why they don't exist:
a) BC^t b) $A^{-1}BC$ c) $A - 3I$ d) D^2

6. Let a simple economy consist of two industries: A and B. The production of $1 worth of A requires 60¢ of A and 10¢ of B. The production of $1 of B requires 70¢ of A and 70¢ of B.
   a) Find the production needed for the external demand of $7000 of A and $2000 of B.
   b) Which, if either, of the two industries is profitable? Justify.
   c) Is the economy productive? Justify.

7. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = 7$. Let B be 3×3 matrix and $\det(B) = -3$. Find:

a) $\det(2B^t)$ b) $\det(BA)$ c) $\det\begin{bmatrix} \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \\ d & e & f \\ g & h & i \end{bmatrix}$ d) $\det\begin{bmatrix} 5a-3d & 5b-3e & 5c-3f \\ d & e & f \\ g & h & i \end{bmatrix}$

8. Let A be a 5×5 matrix with $\det(A)=23$. State whether the following statements are true or false. Justify.
   a) A is invertible.
   b) The row rank of A is 5.
   c) There exists B for which $AX = B$ is inconsistent.
   d) $AX = 0$ can have infinitely many solutions.

9. Let $A=\begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \\ 5 & -2 & 0 \end{bmatrix}$. Find: a) Minor M_{12} b) $adj(A)$ c) $A\cdot adj(A)$ d) $\det(A)$

   e) A^{-1} using $adj(A)$ f) The solution to $AX = B$ using A^{-1}, where $B=\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

10. Given $A=\begin{bmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{bmatrix}$, $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $B=\begin{bmatrix} 0 \\ 0 \\ 11 \\ 0 \end{bmatrix}$

    a) Find $\det(A)$ b) Find $\det(A^{-1})$
    c) Use Cramer's rule to solve $AX = B$ for x_2 **only**.

11. Consider the points $A(5,1,0,2)$, $B(-1,3,5,1)$, $C(2,2,-1,-1)$ and $D(8,0,-6,0)$.
    a) Find the vector $\overrightarrow{AB}$. b) Find the magnitude $|\overrightarrow{AB}|$.
    c) Find a vector equation for the line L that passes through the point C and is parallel to $\overrightarrow{AB}$.
    d) Is the point D on the line L? Justify your answer.

12. Let $\vec{u}_1 = (7,1,2,5)$ and $\vec{u}_2 = (8,1,3,7)$.
    a) If possible, write the vector $\vec{v} = (9,2,-1,0)$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$.
    b) Is the vector $\vec{w} = (3,4,-2,0)$ in the $span\{\vec{u}_1, \vec{u}_2\}$? Explain.

13. Suppose A is a 9x4 matrix
    a) Is it possible for rank of A to equal 9?
    b) If dimension of Col(A) is 2, what is the dimension of Nul(A)?
    c) Is it possible that Dim(Nul(A)) = 0 ?

14. Let the set $S = \{ (x,y,z) \in \Re^3 \mid 2x = 3z \text{ and } y = -z \}$
    a) Is $\vec{0}$ in S? Justify.
    b) Is S closed under multiplication by a scalar? Justify.
    c) Is S closed under addition? Justify.

d) Is S a subspace of $\Re^3$?

15. Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ be the columns of matrix

$$A = \begin{bmatrix} 7 & 2 & 6 & -5 & 9 \\ -1 & -3 & 10 & -2 & -4 \\ 1 & 0 & 2 & -1 & 2 \\ 2 & 1 & 0 & -1 & 3 \end{bmatrix}, \text{ which reduces to } R = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

a) Find the rank and nullity of A.
b) Choose a basis for Col(A) from the columns of A.
c) Express each column of A that is not in your basis as a linear combination of your basis vectors.
d) Find a basis for Nul(A).
e) Which of the following sets are linearly independent? Justify your answer.
i) $\{\vec{a}_1, \vec{a}_3\}$ ii) $\{\vec{a}_2, \vec{a}_3, \vec{a}_4\}$ iii) $\{\vec{a}_2, \vec{a}_3, \vec{a}_5\}$

16. John is carving big and small bears from wood. It takes him 2 hours to make a big bear and 1 hour to make a small bear. Each big bear uses 4 lbs. of wood and each small bear uses 1 lb. of wood. John has 60 hours available for carving and has 80 lbs. of carving wood on hand. He makes a profit of $30 on each big bear and $20 on each small bear. How many big and how many small bears should he make in order to maximize his profit?

Set up the linear programming problem as follows:

a) Define all the variables (using the phrase “the number of”).
b) State the objective and identify the objective function (in terms of the variables).
c) State all the constraints (in terms of the variables).
d) Solve the problem using the graphical method.

17. Use the simplex method to **minimize**:

$$z = 2x_1 + 5x_2 - 2x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &\le 4 \\ -2x_1 - x_2 + x_3 &\le 2 \\ 4x_1 + x_2 + 2x_3 &\le 6 \end{aligned}$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0$$

State the minimum value of z and the corresponding basic feasible solution.

18. Consider the following linear program:

Maximize $z = 2x_1 - x_2 + 7x_3$

subject to

$$\begin{aligned} x_1 - x_2 - x_3 &\le 5 \\ x_1 - 2x_2 + x_3 &\le 4 \end{aligned}$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0$$

Use the simplex algorithm to show that the feasibility region is unbounded. Then find a feasible solution with $P \geq 1000$.

ANSWERS:

1. a) (5, -3) b) inconsistent
2. a) $c - b - 2a = 0$ b) $c - b - 2a \neq 0$
3. a) $9x + 7y - 2z = -3$ b) No
4. a) Let x = # of cups made
   y = # of vases made $3x + 7y + 4z = 178$
   z = # of plates made $2x + 5y + 2z = 122$
   b) $x = 36 - 6t, \quad y = 10 + 2t, \quad z = t$ c) $0 \leq t \leq 6$ if t =1 solution is: (30, 12, 1)
   d) No (no vases y = 0 means $0 = 10 + 2t$, t = - 5 plates, not realistic)
5. a) undefined b) $\begin{bmatrix} -57 & 64 \\ -70 & 68 \end{bmatrix}$ c) $\begin{bmatrix} 0 & -5/2 \\ -1 & -2 \end{bmatrix}$ d) $\begin{bmatrix} -11 & -3 \\ 5 & -14 \end{bmatrix}$
6. a) \$70 000 of A and \$30 000 of B b) A is profitable, B not profitable
   c) economy is productive
7. a) -24 b) -21 c) $\frac{7}{2}$ d) 35
8. . a) true b) true c) false d) false
9. a) -15 b) $\begin{bmatrix} 6 & -2 & -11 \\ 15 & -5 & -2 \\ -18 & -11 & 16 \end{bmatrix}$ c) $\begin{bmatrix} -51 & 0 & 0 \\ 0 & -51 & 0 \\ 0 & 0 & -51 \end{bmatrix}$ d) -51
   e) $-\frac{1}{51}\begin{bmatrix} 6 & -2 & -11 \\ 15 & -5 & -2 \\ -18 & -11 & 16 \end{bmatrix}$ f) $(\frac{19}{51}, -\frac{29}{51}, -\frac{23}{51})$
10. a) -423 b) $-\frac{1}{423}$ c) $\frac{242}{141}$
11. a) $(-6, 2, 5, -1)$ b) $\sqrt{66}$ c) $L: (2, 2, -1, -1) + t(-6, 2, 5, -1)$ d) D is on the line
12. a) $\vec{v} = 7\vec{u}_1 - 5\vec{u}_2$ b) $\vec{w}$ not in the span
13. a) no b) 2 c) yes
14. a) yes b) yes c) yes d) yes
15. a) rank(A) =3 , nullity (A) = 2 b) basis for Col(A) $= \{\vec{a}_1, \vec{a}_2, \vec{a}_5\}$,
    c) $\vec{a}_3 = 2\vec{a}_1 - 4\vec{a}_2, \quad \vec{a}_4 = -\vec{a}_1 + \vec{a}_2$ d) basis for Nul(A) $= \{(-2, 4, 1, 0, 0), (1, -1, 0, 1, 0)\}$,
    e) i) L.I. ii) L.D iii) L.I
16. a) Let x = # of big bears made b) Maximize $P = 30x + 20y$
    y = # of small bears made
    c) $2x + y \leq 60$
    $4x + y \leq 80 \qquad x \geq 0, \quad y \geq 0$
    d) Max P = \$1200 when 0 big bears and 60 small bears are made.
17. Min $z = -\frac{9}{2}$ at $(\frac{1}{4}, 0, \frac{5}{2}, 11, 0, 0)$
18. Parameter $t \geq \frac{972}{13}$, then example of possible solution is z = 1328 at $(0, 100, 204, 309, 0)$ (t =100)