(7) 1. Given the following system:

Let $A = \begin{bmatrix} 1 & 1 & -1 & 8 \\ -4 & -3 & 1 & -26 \\ -5 & -3 & 1 & -30 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

(a) Write the general solution to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

(b) Find the specific solution where $x_1 = 6$.

(c) Write a basis for Nul(A)

(6) 2. Given the following matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & b \\ 0 & 3 & a & b \end{bmatrix}$

(In your answers, use "and" and "or" correctly.)

(a) Under what conditions on a and b is rank(A)=4?

(b) Under what conditions on a and b is rank(A)=3?

(c) Under what conditions on a and b is rank(A)=2?

(5) 3. Let $A = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -3 \\ 0 & 4 & 21 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}$

(a) Find A^{-1}

(b) Solve $A\mathbf{x} = \mathbf{b}$ using your answer to part (a)

(6) 4. Consider the following block matrix, and assume M is invertible, while A, B, C, and D are all square:

$$M = \begin{bmatrix} 0 & B & 0 \\ A & C & 0 \\ 0 & 0 & D \end{bmatrix}$$

(a) Find the block matrix form for M^{-1}

(b) Which submatrices A, B, C, and D must be invertible for M^{-1} to exist?

(4) 5. Set up an augmented matrix for balancing the following chemical equation:

You do not have to solve the system!

___$Ca(OH)_2$ + ___H_3PO_4 → ___$Ca_3(PO_4)_2$ + ___H_2O

(6) 6. Let A be a 4×4 symmetric matrix with $\det(A) = -5$. (Recall that A is symmetric if $A = A^T$.)

For each part, either provide an answer or write "not enough information".

(a) What the value of $\det(-4A^{-1})$?

(b) What is the value of $\det(2A^T - A)$?

(c) What is the value of $\det(A - I)$?

(6) 7. Let A, B, and C be $n \times n$ matrices and suppose $AB^TC^{-1} = I$

(a) Use determinants to explain why A and B must all be invertible

(b) Does A commute with B^TC^{-1}? Why or why not?

(c) Find B^{-1}.

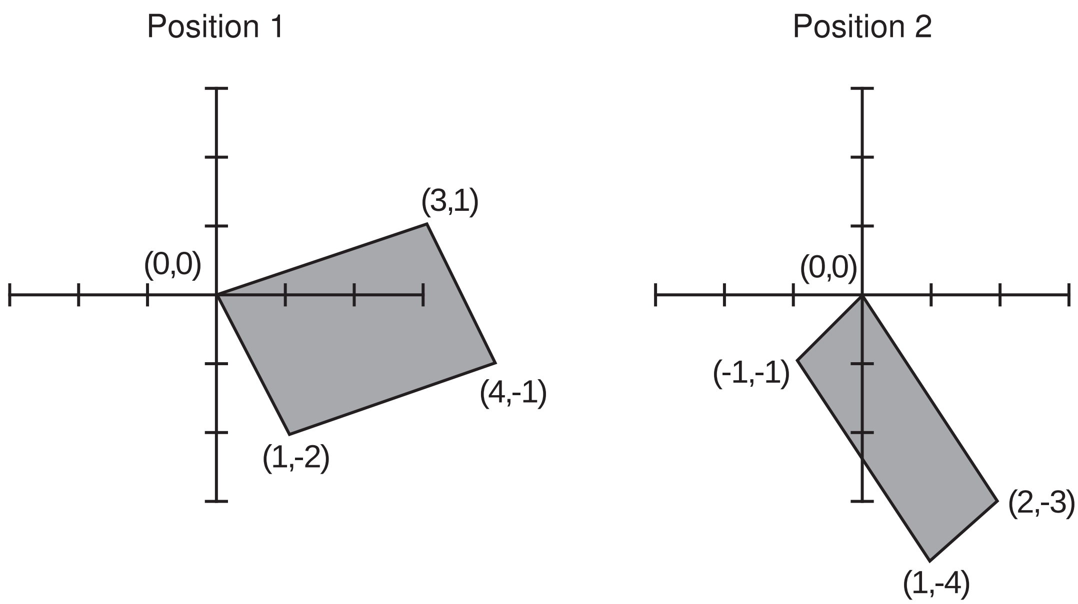
8. Let $A = \begin{bmatrix} 2 & 5 \\ -2 & -8 \\ 8 & 2 \end{bmatrix}$.

(3) Write A as the product LU, where L is lower triangular and U is upper triangular.

(3) 9. Find elementary matrices E_1 and E_2 which satisfy the following equation.

$$E_2E_1 \begin{bmatrix} -5 & 6 \\ 0 & 1 \end{bmatrix} = I$$

(6) 10. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, for some matrix A.
Let T transform the parallelogram in position 1 to the parallelogram in position 2 (as seen below)



(a) Find the area of each parallelogram.

(b) What are the possible values of $\det(A)$?

(c) Give a specific matrix B such that $S(x) = Bx$ will transform the parallelogram in **position 2** into the parallelogram in **position 1**.

(8) 11. Let $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : ad = bc \right\}$ be a subset of $\mathbf{R}^4$.

(a) Does H satisfy closure under vector addition? Justify.

(b) Does H contain the zero vector of $\mathbf{R}^4$? Justify.

(c) Does H satisfy closure under scalar multiplication? Justify.

(d) Is H a subspace of $\mathbf{R}^4$? Justify.

(10) 12. Find a specific example for each of the following, if possible. If not, explain why.

(a) a nonzero 2×2 matrix A such that $\text{Col}(A) = \text{Row}(A)$.

(b) a 2×2 matrix A such that $\text{Nul}(A) = \text{Row}(A)$.

(c) a lower triangular 3×3 matrix A such that A and $A + I$ are both non-invertible.

(d) a square matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ is onto but not 1-1.

(e) A unit vector perpendicular to both $\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(2) 13. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbf{R}^3$, simplify the following expression:

$$\mathbf{u} \cdot [(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})]$$

(10) 14. Given four points $A(2,7,-1)$, $B(3,3,-1)$, $C(3,7,-4)$ and $D(5,5,5)$,

(a) Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find $\text{proj}_{\overrightarrow{AC}}\ \overrightarrow{AB}$ and $\text{perp}_{\overrightarrow{AC}}\ \overrightarrow{AB}$

(c) Find the distance from point B to the line through points A and C

(d) Find the equation of the plane, in normal form, containing points A, B and C

(e) Find the volume of the parallelepiped with edges $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

(6) 15. Let $\mathcal{P}_1$ be the plane $4x - 2y + 5z = 3$, and let $\mathcal{P}_2$ be the plane $-2x + y + kz = 0$.
(Notice that $\mathcal{P}_2$ depends on the coefficient k.)

(a) For what value(s) of k are $\mathcal{P}_1$ and $\mathcal{P}_2$ parallel?

(b) For what value(s) of k are $\mathcal{P}_1$ and $\mathcal{P}_2$ perpendicular?

(c) For what value(s) of k does $(-1,-1,1)$ lie on the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$?

(3) 16. Let A and B be matrices of the same size. Suppose that $\mathbf{x}$ is in both $\text{Nul}(A)$ and $\text{Nul}(B)$.
Show that $\mathbf{x}$ must be in $\text{Nul}(A+B)$.

(4) 17. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ Let $H = \{X : AX = XA\}$.
It is given that H is a subspace of $M_{2\times 2}$.
Find a basis for H.

(5) 18. The following two questions are about vector spaces–not necessarily $\mathbf{R}^n$

(a) Write the definition of a "basis of a vector space", using *25 words or fewer*. Be precise.

(b) Let V and W be vector spaces.
Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V.
Let $T : V \to W$ be a linear transformation such that $T(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathcal{B}$.
Prove that $T(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in V$.