**Answers to NYC exam, fall 2010**

1. (a) The reduced row echelom form of the augmented matrix $[A|\mathbf{b}]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & -1 & 7 \end{bmatrix}.$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Introduce a parameter $t$ for the free variable $x_4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

(b)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 5 \\ -2 \end{bmatrix}.$$

(c)

$$\left\{ \begin{bmatrix} -4 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

2. (a) $a \neq -3$ and $b \neq 3$

(b) Either both $a \neq -3$ and $b = 3$ or both $a = -3$ and $b \neq 3$

(c) $a = -3$ and $b = 3$

3. (a)

$$A^{-1} = \begin{bmatrix} 9 & -26 & -2 \\ 21 & -63 & -5 \\ -4 & 12 & 1 \end{bmatrix}.$$

(b)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15 \\ -40 \\ 8 \end{bmatrix}.$$

4. (a)

$$M^{-1} = \begin{bmatrix} -A^{-1}CB^{-1} & A^{-1} & O \\ B^{-1} & O & O \\ O & O & D^{-1} \end{bmatrix}.$$

(b) $A, B$ and $D$.

5. Introduce variables $x_1, x_2, x_3, x_4$ to balance

$$x_1 \text{ Ca(OH)}_2 + x_2 \text{ H}_3\text{PO}_4 \to x_3 \text{ Ca}_3(\text{PO}_4)_2 + x_4 \text{ H}_2\text{O}.$$

The augmented matrix for the system is

$$\begin{array}{c} \\ \text{Ca} \\ \text{O} \\ \text{H} \\ \text{P} \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & -3 & 0 & 0 \\ 2 & 4 & -8 & -1 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right] \end{array}$$

6. (a)

$$\frac{256}{-5}$$

(b)

$$-5$$

(c) Not enough information

7. (a) $1 = |I_n| = |AB^TC^{-1}| = |A||B||C^{-1}|$, and so $|A| \neq 0$ and $|B| \neq 0$. Therefore $A$ and $B$ are invertible.

(b) Yes, since $B^TC^{-1} = A^{-1}$ and $A$ commutes with its inverse.

(c)

$$B^{-1} = A^T(C^{-1})^T$$

8.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}.$$

9.

$$E_1 = \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & \frac{6}{5} \\ 0 & 1 \end{bmatrix}$$

10. (a) Area of $S$ is 7, area of $T(S)$ is 5.

(b) $\frac{5}{7}$ and $-\frac{5}{7}$

(c) $B = \begin{bmatrix} 0 & -1 \\ \frac{7}{5} & \frac{3}{5} \end{bmatrix}$ ( or $B = \begin{bmatrix} -\frac{8}{5} & -\frac{7}{5} \\ -1 & 0 \end{bmatrix}$)

11. (a) No. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in H$ because $(0)(1) = (0)(0)$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in H$ because $(1)(0) = (0)(0)$,

but $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \notin H$

(b) Yes, because $(0)(0) = (0)(0)$.

(c) Yes. Justification: Suppose $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in H$ and $k \in \mathbb{R}$. Since $ad = bc$ and $k\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} =$ $\begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix}$, we have $(ka)(kd) = k^2(ad) = k^2(bc) = (kb)(kc)$, and therefore $k\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in H$.

(d) No, since $H$ does not satisfy closure under addition.

12. (a) Any invertible matrix, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) This is not possible. Each vector in Nul($A$) is orthogonal to each vector in Row($A$), thus if a vector is in both Nul($A$) and Row($A$) it is orthogonal to itself and so can only be the zero vector. This implies $\text{Nul}(A) = \text{Row}(A) = \{\mathbf{0}\}$ and thus $\dim(\text{Nul}(A)) + \dim(\text{Row}(A)) = 0$. However the domain space for the linear transformation arising from multiplying vectors by $A$ is $\mathbb{R}^2$ and so $\dim(\text{Nul}(A)) + \dim(\text{Row}(A)) = \dim(\mathbb{R}^2) = 2$. A contradiction.

(c)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) This is not possible. If $T$ is onto $A$ has a pivot position in every row. But then since $A$ is square is should have a pivot position in every column, which implies $T$ is one to one. (You can also give a dimension argument as in part (b).)

(e)

$$\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

13. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

14. (a)

$$\frac{1}{\sqrt{170}}$$

(b) $\text{proj}_{\overrightarrow{AC}}\, \overrightarrow{AB} = \begin{bmatrix} 1/10 \\ 0 \\ -3/10 \end{bmatrix}$ $\quad$ $\text{perp}_{\overrightarrow{AC}}\, \overrightarrow{AB} = \begin{bmatrix} 9/10 \\ -4 \\ 3/10 \end{bmatrix}$ $\quad$ (C) $\frac{\sqrt{1690}}{10} = \frac{13\sqrt{10}}{10}$

(d) $12\,x + 3\,y + 4\,z = 41$

(e) 54

15. (a) $k = -\frac{5}{2}$

(b) $k = 2$

(c) $k = -1$

16. Since $\mathbf{x}$ is in both Nul$(A)$ and Nul$(B)$, we have $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$. Therefore $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, and thus $\mathbf{x} \in \text{Nul}(A + B)$.

17.

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$$

18. (a) A basis of a vector space is a set of linearly independent vectors that spans the vector space.

(b) For any $\mathbf{x} \in V$ there are scalars $c_1, c_2, \cdots c_n$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

since $\mathcal{B}$ spans $V$, and therefore

$$\begin{aligned} T(\mathbf{x}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \quad \text{since } T \text{ is a linear transformation} \\ &= c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_n\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

END of EXAM