1.)[**8 marks**] Solve the following systems, or show that they are inconsistent.

a) $\begin{cases} 2x_1 - 4x_2 + 16x_3 - 4x_4 = 11 \\ 2x_1 - 6x_2 + 22x_3 - 6x_4 = 10 \\ -2x_1 + 5x_2 - 19x_3 + 5x_4 = -8 \end{cases}$

b) $\begin{cases} x_1 + 3x_2 + 5x_3 = -24 \\ 2x_1 + 7x_2 + 11x_3 = -54 \\ -x_1 + x_2 = -6 \end{cases}$

2.)[**4 marks**] Consider the system $\begin{cases} x_1 + 3x_2 + 5x_3 = 2 \\ -x_1 - 2x_2 - 3x_3 = k \\ -3x_1 - 8x_2 - 13x_3 = -5 \end{cases}$

For which value(s) of k is the system

i) consistent?

ii) inconsistent?

3.)[**5 marks**] Find an equation of the plane that passes through the points $P(-4,-1,-1)$, $Q(-2,0,1)$ and $R(-1,-2,-3)$.

4.)[**5 marks**] The *Nuts 'R' Us* company produces 3 types of trail mix, using peanuts and almonds (among others). Trail mix I requires 100g of peanuts and 100g of almonds. Trail mix II requires 100g of peanuts and 200g of almonds. Trail mix III requires 300g of peanuts and 100g of almonds. The company has 1.6kg of peanuts and 1.5kg of almonds available, and wants to know how many packages of each mix can be produced, using all the supplies.

(**Note:** 1kg = 1000g)

a) Define the necessary variables, and set up a system of equations in order to solve the problem.

b) Knowing that $(17-5t, -1+2t, t)$, $t \in \mathbb{R}$, is a general solution to the problem, find all realistic solutions if the company wants only **complete packages**.

5.)[**7 marks**] Consider the matrices $A = \begin{bmatrix} 1 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, linebreak

$C = \begin{bmatrix} 0 & 3 & -3 \\ -1 & 2 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. Find (if possible):

a) $(A^T + C)D$ **b)** $(AB)^T - C$ **c)** $B^{-1}C$ **d)** $B(A + C^T)$

6.)[**8 marks**] An economy consists of two industries: Steel and Coal. The production of \$1 of steel requires \$0.50 of steel and \$0.30 of coal. The production of \$1 of coal requires \$0.40 of steel and \$0.60 of coal. There is an external demand for \$8000 of steel and \$12000 of coal.

a) How much should each industry produce to satisfy the external demand?

b) What is the internal consumption?

c) Is each industry profitable? Justify your answer.

d) Is the economy productive? Justify your answer.

7.)[**8 marks**] Consider the matrices $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \\ -10 \end{bmatrix}$.

a) Find $\det(A)$.

b) Find $\operatorname{adj}(A)$.

c) Use **a)** and **b)** to find A^{-1}.

d) Use A^{-1} to solve the system $AX = B$.

8.)[**6 marks**] Given the system of equations

$$\begin{cases} 2x_1 - 5x_2 + 2x_3 + 3x_4 = 0 \\ x_1 + 4x_2 + 5x_3 - 2x_4 = 0 \\ 3x_1 + 7x_2 - 11x_3 + 3x_4 = 3 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 = 0 \end{cases},$$

and knowing that $\det \begin{bmatrix} 2 & -5 & 2 & 3 \\ 1 & 4 & 5 & -2 \\ 3 & 7 & -11 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix} =$ -1288, use Cramer's rule to find x_3.

9.)[**4 marks**] Let A, B and C be 4×4 matrices such that $\det(A) = 4$, $\det(B) =$ -2 and $\det(C) = 7$.

a) Find $\det(BA)$.

b) Find $\det(2C)$.

c) Find $\det(A^2 B^T C^{-1})$.

d) Find $\operatorname{rank}(A)$.

10.)[4 marks] Given the points $P_1(3,2)$, $P_2(-5,4)$, $P_3(1,-3)$ and $P_4(-3,-2)$:

a) Find the magnitude of the vector $\overrightarrow{P_1P_2}$.

b) Find a vector equation of the line parallel to $\overrightarrow{P_1P_2}$ that passes through P_3.

c) Is P_4 on the line found in **b)**? Justify your answer.

11.)[4 marks] Consider the set of vectors $S = \{(x\ ,\ y\ ,\ -3x+2y)\ :\ x, y \in \mathbb{R}\}$.

a) Is $\overrightarrow{0}$ in S? Justify.

b) Is S closed under addition? Justify.

c) Is S close under scalar multiplication? Justify.

d) Is S a subspace of $\mathbb{R}^3$? Justify.

12.)[5 marks] Given $\overrightarrow{u} = (1,4,-7)$, $\overrightarrow{v} = (3,0,3)$ and $\overrightarrow{w} = (2,1,k)$.

a) Find the value(s) of k so that $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is a linearly dependent set.

b) Find the value(s) of k so that $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is a basis of $\mathbb{R}^3$.

13.)[3 marks] True or False? Justify your answer.

a) Suppose $W_1 = \text{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ and $W_2 = \text{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$.
If $\dim(W_2) = 4$, then $\dim(W_1) = 3$.

b) If A is a 3×5 matrix, then the nullity of A is at least 2.

14.)[8 marks] Given that the matrix $A = \begin{bmatrix} 2 & 7 & -5 & -3 & 13 \\ 1 & 0 & 1 & 4 & 3 \\ 1 & 3 & -2 & -2 & 6 \\ 5 & 2 & 3 & 1 & 17 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$:

a) Find the rank and nullity of A.

b) Find a basis for the null space of A.

c) Find a basis for the column space of A.

d) Express each of the columns of A not in the basis of $\mathrm{Col}(A)$ as a linear combination of the basis vectors.

e) Let the columns of A be denoted by $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, $\overrightarrow{a_3}$, $\overrightarrow{a_4}$ and $\overrightarrow{a_5}$. Which of the following sets are linearly dependent, which ones are linearly independent? (Justify)

i) $\{\overrightarrow{a_1}, \overrightarrow{a_4}\}$

ii) $\{\overrightarrow{a_1}, \overrightarrow{a_3}, \overrightarrow{a_4}\}$.

iii) $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \overrightarrow{a_4}, \overrightarrow{a_5}\}$.

15.)[**7 marks**] Use the simplex method to show that the following problem has no solution. Also, find (x, y, z) that satisfy the constraints and for either $P = -1816$ or $P < -1816$.

Minimize $P = 3x - 4y - 2z$

$$\text{subject to} \begin{cases} 3x - 4y \leqslant 12 \\ x + y - 4z \leqslant 4 \\ 4x - 2y + 5z \leqslant 20 \\ x \geqslant 0 \quad , \quad y \geqslant 0 \quad , \quad z \geqslant 0 \end{cases}$$

16.)[**7 marks**] Use the simplex method to maximize P. Your solution should include the maximum value and the corresponding feasible solution.

$$P = 4x + 3y - 8z, \qquad \text{subject to} \begin{cases} x - 3y + 2z \leqslant 3 \\ -2y + z \leqslant 1 \\ x + y - 2z \leqslant 7 \\ x \geqslant 0 \quad , \quad y \geqslant 0 \quad , \quad z \geqslant 0 \end{cases}$$

17.)[**7 marks**] Farmer Pete has 300 acres on which he plants all his crops. Among those crops are corn and wheat. Corn requires 5-worker days and $15 of capital for each acre planted. Wheat requires 2-worker days and $20 of capital for each acre planted. Corn yields $40 in revenue per acre, while wheat yields $30 in revenue per acre. Farmer Pete estimates that he has at most $4000 of capital and 750-worker days of labour available for the year. What planting strategy will maximize Farmer Pete's revenue?

Set up the linear programming problem as follows:

a) Define all the necessary variables.

b) State the objective, and identify the objective function (in terms of the variables).

c) State all the constraints (in terms of the variables).

d) Solve the problem using the graphical method.

ANSWERS

1.) **a)** Inconsistent **b)** (6,0,-6)

2.) **i)** $k = -1$ **ii)** $k \neq -1$

3.) $2y - z = -1$

4.) **a)** Let
$$\begin{aligned} x_1 &= \text{number of packagesof trail mix I} \\ x_2 &= \text{number of packagesof trail mix II} \\ x_3 &= \text{number of packagesof trail mix III} \end{aligned}$$

$$\left\{ \begin{array}{rcrcrcl} 100x_1 &+& 100x_2 &+& 300x_3 &=& 1600 \\ 100x_1 &+& 200x_2 &+& 100x_3 &=& 1500 \end{array} \right.$$

b) t=1 (12,1,1); t=2 (7,3,2); t=3 (2,5,3).

5.) **a)** $\begin{bmatrix} -10 \\ 15 \end{bmatrix}$ **b)** $\begin{bmatrix} 10 & -2 & 5 \\ -9 & 10 & -3 \end{bmatrix}$ **c)** $\frac{1}{10}\begin{bmatrix} 2 & 8 & -26 \\ -1 & 11 & -2 \end{bmatrix}$ **d)** Undefined.

6.) **a)** \$100000 Steel, \$105000 Coal **b)** \$92000 Steel, \$93000 Coal **c)** Only Steel is profitable, sum of column < 1 **d)** Yes, $(I - C)^{-1} \geqslant 0$.

7.) **a)** -5 **b)** $\begin{bmatrix} 6 & 5 & -4 \\ -3 & -5 & 2 \\ -7 & -5 & 3 \end{bmatrix}$ **c)** $\begin{bmatrix} \frac{-6}{5} & -1 & \frac{4}{5} \\ \frac{3}{5} & 1 & \frac{-2}{5} \\ \frac{7}{5} & 1 & \frac{-3}{5} \end{bmatrix}$ **d)** $\begin{bmatrix} -17 \\ 10 \\ 16 \end{bmatrix}$

8.) $x_3 = -\dfrac{57}{322}$

9.) **a)** -8 **b)** 112 **c)** $-\dfrac{32}{7}$ **d)** 4

10.) **a)** $\sqrt{68}$ **b)** $\begin{bmatrix} 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -8 \\ 2 \end{bmatrix}$ **c)** Yes (t=1/2)

11.) **a)** Yes **b)** Yes **c)** Yes **d)** Yes **12.)** **a)** $k = 0$ **b)** $k \neq 0$

13.) **a)** True **b)** True

14.) **a)** Rank = 3, Nullity = 2 **b)** $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

c) $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ and $\overrightarrow{a_4}$. **d)** $\overrightarrow{a_1} = 1\overrightarrow{a_1} + 0\overrightarrow{a_2} + 0\overrightarrow{a_4}$ $\overrightarrow{a_2} = 0\overrightarrow{a_1} + 1\overrightarrow{a_2} + 0\overrightarrow{a_4}$
$\overrightarrow{a_3} = 1\overrightarrow{a_1} - 1\overrightarrow{a_2} + 0\overrightarrow{a_4}$ $\overrightarrow{a_4} = 0\overrightarrow{a_1} + 0\overrightarrow{a_2} + 1\overrightarrow{a_4}$ $\overrightarrow{a_5} = 3\overrightarrow{a_1} + 1\overrightarrow{a_2} + 0\overrightarrow{a_4}$
e) **i)** LI **ii)** LI **iii)** LD

15.) $P = -1816$ when $x_1 = 0$, $x_2 = 404$ and $x_3 = 100$ ($s_1 = 1628$, $s_2 = 0$ and $s_3 = 328$)

16.) $P = 27$ when $x_1 = 6$, $x_2 = 1$ and $x_3 = 0$ ($s_1 = 0$, $s_2 = 3$ and $s_3 = 0$)

17.) **a)** x = #acres of corn, y = #acres of wheat

b) Maximize the revenue: $R = 40x + 30y$

c Constraints: $\begin{cases} x + y \leqslant 300 \\ 5x + 2y \leqslant 750 \\ 15x + 20y \leqslant 4000 \\ x \geqslant 0 \quad y \geqslant 0 \end{cases}$

d) $R = \$7750$ when $x = 100$ and $y = 125$