(Marks)

1. Given the matrix $A = \begin{bmatrix} 1 & 4 & -1 & 1 \\ -2 & -8 & -2 & 6 \\ 3 & 12 & -1 & -1 \end{bmatrix}$:
   (a) Solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$.
   (b) Solve the system $A\mathbf{x} = \mathbf{0}$.

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the rule
   $$T(x_1, x_2, x_3, x_4) = (2x_1, x_2 + x_3, 2x_3 - x_4).$$
   (a) Find the standard matrix for T.
   (b) Is T *one-to-one*? Justify.
   (c) Does T map $\mathbb{R}^4$ *onto* $\mathbb{R}^3$? Justify.

3. Given the system of linear equations:
   $$\begin{aligned} x + y - z &= 0 \\ x + (k+1)y + 2z &= 0 \\ x + y + (k-5)z &= 0 \end{aligned}$$
   find all values of k (if any) for which the system has:
   a) no solutions b) a unique solution c) infinitely many solutions

4. Find a second degree polynomial whose curve contains the points $(1, 1)$ and $(2, 6)$ and whose derivative at $x = 1$ is 4.

5. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$ find A^{-1}.

6. Find an LU-factorization for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix}$.

7. Let A be the block matrix $A = \begin{bmatrix} I & M \\ N & 0 \end{bmatrix}$ where M and N are $n \times n$ invertible matrices. Find the block form of A^{-1}.

8. If $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, it is given that $A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$
   Use A^{-1} to find $(AA^T)^{-1}$.

9. Given $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = 5$
   Find:
   (a) $|4A|$
   (b) $|AA^T|$
   (c) $|\mathrm{adj}(A)|$
   (d) $\begin{vmatrix} a & b & c \\ g+3a & h+3b & i+3c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \end{vmatrix}$

10. Suppose A, B and C are $n \times n$ matrices and $ABC = I$. Find B^{-1}.

11. If A is a 9×9 matrix such that $A^T = -A$, then prove that $\det(A) = 0$. Is the same result true for a 10×10 matrix A? Why or why not?

12. Given $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 5 & 2 & -7 & 3 \\ 3 & 0 & 6 & 2 \\ 5 & 2 & -4 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
    (a) Solve only for x_3, using Cramer's Rule.
    (b) How many solutions does $A\mathbf{x} = 0$ have?

13. Are the following true or false. (All matrices are $n \times n$.) Justify your answer. No credit will be given without justification.
    (a) $|E_1E_2| \neq 0$, where E_1 and E_2 are elementary matrices.
    (b) $(A + B)(A - B) = A^2 - B^2$
    (c) $|A + I| = |A| + 1$
    (d) If S and T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that S is onto but T is NOT onto, then the composition of the transformations $S \circ T$ is NOT onto.
    (e) The nonpivot columns of a matrix form a linearly dependent set.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which scales every vector by the scalar 5 then reflects the vector through the y-axis and finally rotates the vector by $\pi/4$ radians clockwise. Which of the following matrices is the standard matrix for T? Circle your answer. No justification is required for this question.
    a) $\frac{5}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ b) $\frac{5}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ c) $\frac{5}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$
    d) $\frac{5}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ e) $\frac{5}{\sqrt{2}}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ f) $\frac{5}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(Marks)

15. Let A be a 7×9 matrix with $\text{rank}(A) = 4$.
    (a) What is $\dim(\text{Nul}(A))$?
    (b) What is $\dim(\text{Row}(A))$?
    (c) What is $\text{rank}(A^T)$?
    (d) What is $\dim(\text{Nul}(A^T))$?

16. Given $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
    (a) Find a basis for column space of A and state its dimension.
    (b) Write every column of A not in this basis as a linear combination of the basis vectors.
    (c) Find a basis for null space of A and state its dimension.
    (d) Find a basis for row space of A.
    (e) Do the columns of A span $\mathbb{R}^4$?

17. Find a basis for each of the following vector spaces S. State the dimension of the vector space in each case.
    (a) $S = \{\text{all } 2 \times 2 \text{ matrices } A \text{ such that } A^T = A\}$.
    (b) $S = \{\text{all vectors in } \mathbb{R}^3 \text{ orthogonal to } \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\}$.
    (c) $S = \{\text{all polynomials } p(x) \text{ in } \mathbb{P}_3 \text{ such that } p(0) = 0\}$.

18. Given the subset $S = \{\text{all } \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \text{ such that } x \leq 0 \text{ and } y \leq 0\}$ of $\mathbb{R}^2$, answer the following:
    (a) Does S contain the zero vector?
    (b) Is S closed under scalar multiplication?
    (c) Is S closed under vector addition?
    (d) Is S a subspace of $\mathbb{R}^2$?

19. For each of the following sets S, determine if it is a subspace of the given vector space. Justify your answer.
    (a) $S = \{\text{all } 3 \times 3 \text{ matrices } A \text{ such that } |A| = 0\}$ in $M_{3\times 3}$.
    (b) $S = \{\text{all } 2 \times 3 \text{ matrices } X \text{ such that } \begin{bmatrix} 1 & 2 \\ 8 & 16 \end{bmatrix} X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}$ in $M_{2\times 3}$.

20. Let $\mathcal{P}$ be the plane defined by the equation $x - 2y + 2z = 0$.
    (a) Find a basis for the intersection of $\mathcal{P}$ and the xy plane.
    (b) Find the intersection of $\mathcal{P}$ with the line $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
    (c) Find the distance from the point $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ to $\mathcal{P}$.
    (d) Give a parametric vector equation for the line through the origin which intersects $\mathcal{P}$ at a right angle.

21. Consider the three points $A(1, 2, 3)$, $B(2, 3, 1)$ and $C(3, 1, 2)$.
    (a) Find the area of the triangle containing A, B, and C.
    (b) Find the equation of a plane containing A, B, and C.
    (c) Find the distance from the point A to the line through points B and C.

22. In $\mathbb{R}^3$, find $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) + \mathbf{i} \times (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \times (\mathbf{i} \times \mathbf{i})$.

23. Write $A = \begin{bmatrix} 4 & 3 \\ 1 & 0 \end{bmatrix}$ as the product of elementary matrices.

24. Let $T : V \to W$ be a linear transformation from the vector space V to the vector space W. Let $\{v_1, v_2, v_3\}$ be a linearly dependent set in V.
    Prove that $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set in W.

25. Let $S = \{v_1, v_2, v_3\}$ be a set of linearly independent vectors in a vector space V.
    Prove that the set $\mathcal{C} = \{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is also linearly independent.