1. Given $f(x) = 2x \arctan 2x - \frac{1}{2}\log(1+4x^2) + \arcsin \frac{2}{3}$.

a. Find $f'(x)$ and simplify your answer. b. Evaluate $f'\left(\frac{1}{2}\right)$

2. Evaluate each of the following limits, using ∞ and $-\infty$ when appropriate.

a. $\displaystyle\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{2x}$ b. $\displaystyle\lim_{x\to 0^+}\left(e^{-2/x}\log x\right)$ c. $\displaystyle\lim_{x\to 0}\frac{e^{6x}-6x-1}{x^2}$

3. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{2x+1}{\sqrt{x-3}}\,dx$ b. $\displaystyle\int \frac{9x-1}{(x-3)(x^2+4)}\,dx$

c. $\displaystyle\int x \operatorname{arcsec} x\,dx$ d. $\displaystyle\int_0^{\frac{1}{4}\pi}\sin^3 2x\cos^4 2x\,dx$

e. $\displaystyle\int_0^{\frac{1}{2}}\frac{\arcsin x}{\sqrt{1-x^2}}\,dx$ f. $\displaystyle\int e^{3x}\sin x\,dx$ g. $\displaystyle\int\frac{dx}{\sqrt{9x^2-16}}$

4. Evaluate each of the following improper integrals.

a. $\displaystyle\int_1^{\frac{2}{3}\sqrt{3}}\frac{dx}{x\sqrt{x^2-1}}$ b. $\displaystyle\int_4^{\infty}\frac{dx}{x\log x}$

5. Solve the differential equation

$$2y\frac{dy}{dx} = y^2 - 1; \quad y(0) = 2.$$

6. Sketch the region enclosed by $y = 2/x - 1$ and $y = 2 - x$, and find its area.

7. Let $\mathscr{R}$ be the region enclosed by $y = \sin x^2$ and the x-axis on $[\,0, \sqrt{\pi}\,]$.

a. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis.

b. Set up, but do not evaluate, an integral that represents the volume of the solid obtained by revolving $\mathscr{R}$ about the x-axis.

8. Determine whether the sequence converges or diverges; if it converges, find its limit.

a. $\left\{1 + \cos\frac{1}{2}(2n+1)\pi\right\}$ b. $\left\{(-1)^n\dfrac{3n^2+n-2}{n^2}\right\}$

9. Determine whether each statement is true or false. Justify each answer, with a proof or a counterexample, as appropriate).

a. If $\lim|a_n| \neq 0$ then $\lim a_n \neq 0$.

b. If $\lim a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty}\sin a_n$ converges.

10. Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{3^{n+1}+2^n}{4^n}.$$

11. Classify each of the following series as convergent or divergent, and justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$ b. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n-e}{n^2}\right)^{2n}$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^3-1}}{n^2+1}$ d. $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}$

12. Classify each of the following series as absolutely convergent, conditionally convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{n^3+1}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos\frac{1}{n}$ c. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$

13. Determine the radius and interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{3^{n-1}(x+1)^n}{n\sqrt{n+1}}.$$

14. Let $f(x) = \log(1+x)$.

a. Write the first five non-zero terms of the Maclaurin series of f.

b. Find a formula for the k^{th} term of the Maclaurin series, and write the series using sigma notation.

Solution outlines

1. a. $f'(x) = 2\arctan 2x + \dfrac{4x}{1+4x^2} - \dfrac{4x}{1+4x^2} + 0 = 2\arctan 2x$.

b. $f'\left(\frac{1}{2}\right) = 2\arctan 1 = \frac{1}{2}\pi$.

2. a. One application of l'Hôpital's rule gives

$$\lim_{x\to\infty}\left\{2x\log(1+4/x)\right\} = 2\lim_{t\to 0^+}\frac{\log(1+4t)}{t} = 8\lim_{t\to 0^+}\frac{1}{1+4t} = 8,$$

where $t = 1/x$, so the limit in question is equal to e^8.

b. One application of l'Hôpital's rule, after letting $t = 1/x$, gives

$$\lim_{x\to 0^+}\left(e^{-2/x}\log x\right) = -\lim_{t\to\infty}\frac{\log t}{e^{2t}} = -\lim_{t\to\infty}\frac{1}{2te^{2t}} = 0.$$

c. Two applications of l'Hôpital's rule gives

$$\lim_{x\to 0}\frac{e^{6x}-6x-1}{x^2} = 3\lim_{x\to 0}\frac{e^{6x}-1}{x} = 18\lim_{x\to 0}e^{6x} = 18.$$

3. a. Repeated partial integration (integrating the the fractional power and differentiating the polynomial) gives

$$\int\frac{2x+1}{\sqrt{x-3}}\,dx = 2(2x+1)\sqrt{x-3} - \tfrac{8}{3}(x-3)^{3/2} + C$$
$$= \tfrac{2}{3}(2x+15)\sqrt{x-3} + C.$$

b. Resolving the integrand into partial fractions and then integrating term by term yields

$$\int\frac{9x-1}{(x-3)(x^2+4)}\,dx = \int\left\{\frac{2}{x-3} - \frac{2x-3}{x^2+4}\right\}dx$$
$$= \log\frac{(x-3)^2}{x^2+4} + \tfrac{3}{2}\arctan\tfrac{1}{2}x + C.$$

c. Partial integration gives

$$\int x\operatorname{arcsec} x\,dx = \tfrac{1}{2}x^2\operatorname{arcsec} x - \tfrac{1}{2}\int\frac{x}{\sqrt{x^2-1}}\,dx$$
$$= \tfrac{1}{2}x^2\operatorname{arcsec} x - \tfrac{1}{2}\sqrt{x^2-1} + C.$$

d. Changing the variable of integration to $t = \cos(2x)$ gives

$$\int_0^{\frac{1}{4}\pi}\sin^3 2x\cos^4 2x\,dx = \tfrac{1}{2}\int_0^1 t^4(1-t^2)\,dt = \tfrac{1}{70}t^5(7-5t^2)\Big|_0^1 = \tfrac{1}{35}.$$

e. Changing the variable of integration to $t = \arcsin x$ gives

$$\int_0^{\frac{1}{2}}\frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{1}{6}\pi}t\,dt = \tfrac{1}{2}t^2\Big|_0^{\frac{1}{6}\pi} = \tfrac{1}{72}\pi^2.$$

f. Repeated partial integration (integrating the trigonometric function and differentiating the exponential function) gives

$$\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3e^{3x}\sin x - 9\int e^{3x}\sin x\,dx,$$

and therefore

$$\int e^{3x}\sin x\,dx = \tfrac{1}{10}e^{3x}(3\sin x - \cos x) + C.$$

g. Applying a standard integral formula gives

$$\int\frac{dx}{\sqrt{9x^2-16}} = \tfrac{1}{3}\log\left|3x + \sqrt{9x^2-16}\,\right| + C.$$

4. a. A standard integral formula gives

$$\int_1^{\frac{2}{3}\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} \tfrac{2}{3}\sqrt{3} - \operatorname{arcsec} 1 = \tfrac{1}{6}\pi,$$

since arcsec is continuous on $[\,1, \frac{2}{3}\sqrt{3}\,]$.

b. One has

$$\int_4^{\infty} \frac{dx}{x\log x} = \lim_{t\to\infty} \log\log t - \log\log 4 = \infty$$

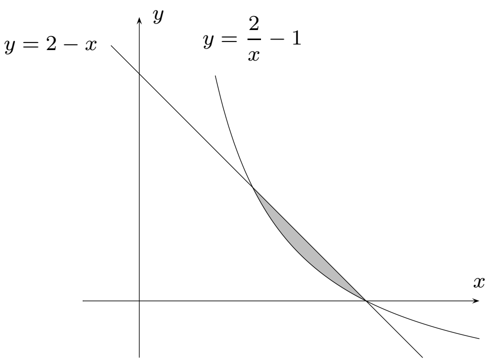
(so the integral diverges).

5. Separating variables and integrating gives

$$\int \frac{2y}{y^2-1}\,dy = \int dx, \quad \text{or} \quad \log|y^2-1| = x + C, \quad \textit{i.e.,} \quad y^2 = Ae^x + 1,$$

where $A = \pm e^C$. Now $y(0) = 2$ gives $A = 3$ and $y > 1$, and so $y = \sqrt{3e^x+1}$.

6. Below is a sketch of the region in question.



The curves meet where $2 - x = 2/x - 1$, or $0 = x^2 - 3x + 2 = (x-1)(x-2)$, *i.e.*, where $x = 1$ or $x = 2$. On $(\,1, 2\,)$ the line is above the hyperbola, so the area of the region in question is

$$\begin{aligned}\int_1^2 \big\{(2-x) - (2/x - 1)\big\}\,dx &= \int_1^2 (3 - x - 2/x)\,dx \\ &= (3x - \tfrac{1}{2}x^2 - 2\log x)\,\Big|_1^2 \\ &= \tfrac{3}{2} - 2\log 2.\end{aligned}$$

7. a. The solid obtained by revolving $\mathscr{R}$ about the y-axis can be decomposed into cylindrical shells of radius x and height $\sin x^2$, for $0 \leqslant x \leqslant \sqrt{\pi}$, so its volume is equal to

$$2\pi \int_0^{\sqrt{\pi}} \sin x^2\,dx = -\pi\cos x^2\,\Big|_0^{\sqrt{\pi}} = 2\pi.$$

b. The solid obtained by revolving $\mathscr{R}$ about the x-axis can be decomposed into disks of radius $\sin x^2$, for $0 \leqslant x \leqslant \sqrt{\pi}$, so its volume is represented by the integral

$$\pi \int_0^{\sqrt{\pi}} \sin^2 x^2\,dx.$$

8. a. Since $\cos \frac{1}{2}(2n+1)\pi = 0$ for every natural number n, the given sequence converges to 1 (each of its terms is equal to 1).

b. Let a_n denote the general term of the given sequence. Since $\lim a_{2n} = 3$ and $\lim a_{2n+1} = -3$, it follows that $\{a_n\}$ has no limit.

9. a. This statement is true. For if $\lim|a_n| \neq 0$, there is a positive real number ε_0 such that for any natural number N there is a natural number $n \geqslant N$ for which $\big||a_n| - 0\big| \geqslant \varepsilon_0$, *i.e.*, $|a_n - 0| \geqslant \varepsilon_0$, which means that $\lim a_n \neq 0$ by definition.

b. This statement is false. For example,

$$a_n = \arcsin\frac{1}{n} \to 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \sin a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges.

10. The given series is the sum of two geometric series, and in fact

$$\sum_{n=0}^{\infty} \frac{3^{n+1} + 2^n}{4^n} = \sum_{n=0}^{\infty} 3\big(\tfrac{3}{4}\big)^n + \sum_{n=0}^{\infty} \big(\tfrac{1}{2}\big)^n = \frac{3}{1-\frac{3}{4}} + \frac{1}{1-\frac{1}{2}} = 14.$$

11. a. The given series diverges because it is the difference of a divergent ($p = 1$) and a convergent ($p = 2$) p-series. (Alternatively, the given series diverges with the harmonic series because its terms are larger than $\frac{3}{4}n^{-1}$ if $n > 2$.)

b. Since

$$\lim \sqrt[n]{\left|\frac{2n-e}{n^2}\right|^{2n}} = \lim \frac{(2-e/n)^2}{n^2} = 0,$$

the series converges by the Root Test.

c. If $n > 1$ then $n^3 - 1 > \frac{1}{4}n^3$, $n^2 + 1 < 2n^2$, and therefore

$$\frac{\sqrt{n^3-1}}{n^2+1} > \tfrac{1}{4}n^{-1/2};$$

so the series in question diverges with $\sum n^{-1/2}$ ($p = \frac{1}{2}$) by the Comparison Test. (Alternatively, the Limit Comparison Test could be used.)

d. Since

$$\frac{(n!)^2}{(2n)!} = \frac{1}{2^n} \cdot \frac{n(n-1)\cdots 2\cdot 1}{(2n-1)(2n-3)\cdots 3\cdot 1} \leqslant \frac{1}{2^n},$$

the given series converges by the Comparison Test. (Alternatively, the Ratio test could be used.)

12. a. Since

$$0 < \frac{\arctan n}{n^3+1} < \tfrac{1}{2}\pi n^{-3}, \quad \text{for} \quad n > 0,$$

the series in question is absolutely convergent by the Comparison Test.

b. Since $\lim \cos\frac{1}{n} = 1$, the series in question diverges by the vanishing criterion.

c. Let $a_n = n/(n^2+1)$. If $n > 1$ then $a_n > \frac{1}{2}n^{-1}$, and so $\sum(-1)^n a_n$ is not absolutely convergent by the Comparison Test. However, $a_n > 0$, $\{a_n\}$ is decreasing since

$$\frac{d}{dx}\left\{\frac{x}{x^2+1}\right\} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{if} \quad x > 1,$$

and

$$\lim a_n = \lim \frac{1}{n} \cdot \frac{1}{1 + 1/n^2} = 0,$$

so $\sum(-1)^n a_n$ converges by the Alternating Series Test. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

13. Let u_n denote the general term of the series in question. Then

$$\lim\left|\frac{u_{n+1}}{u_n}\right| = \frac{3|x+1|}{\sqrt{(1+1/n)(1+2/n)}} = 3|x+1|,$$

so $\sum u_n$ is absolutely convergent if $|x+1| < \frac{1}{3}$, *i.e.*, $-\frac{4}{3} < x < -\frac{2}{3}$, by the Ratio Test. This means that the radius of convergence of $\sum u_n$ is $\frac{1}{3}$. If $x = -\frac{4}{3}$ or $x = -\frac{2}{3}$ then

$$|u_n| = \frac{1}{n\sqrt{n+1}} < n^{-3/2},$$

and so $\sum u_n$ is (absolutely) convergent by the Comparison Test. Therefore, the interval of convergence of $\sum u_n$ is $\big[-\frac{4}{3}, -\frac{2}{3}\big]$.

14. We have

$$\begin{aligned}f(x) = \log(1+x) &= \int_0^x \frac{dt}{1+t} = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^k\,dt \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\,x^k \qquad \text{(b.)} \\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots \qquad \text{(a.)}\end{aligned}$$