[10] 1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 & -1 & 5 & 11 \\ 5 & 10 & 5 & 10 & -5 & 25 & 55 \\ -2 & -4 & -1 & -1 & -1 & 0 & 2 \\ 1 & 2 & 3 & 8 & -9 & 31 & 75 \\ 3 & 6 & 1 & 0 & 2 & 0 & -5 \end{bmatrix}$, which row reduces to

$$B = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the rank of A.

(b) Find a basis for Col(A). What is its dimension?

(c) Write the 6th and 7th column of A as a linear combination of the vectors obtained in (b).

(d) Find a basis for Nul(A). What is its dimension?

(e) Find a basis for Row(A).

(f) Write the first row of A as a linear combination of the vectors obtained in (e).

(g) What is the dimension of Nul(A^T)?

Answers:
1.
(a) $rankA = 4$
(b) A basis for Col(A) is
$\{(1,5,-2,1,3),(1,5,-1,3,1),(-1,-5,-1,-9,2),(5,25,0,31,0)\}$. $dim(\text{Col}(A)) = 4$.
(c)$(5,25,0,31,0) = 0(1,5,-2,1,3)+0(1,5,-1,3,1)+0(-1,-5,-1,-9,2)+1(5,25,0,31,0)\}$.
$(11,55,2,75,-5) = 2(1,5,-2,1,3) + (-1)(1,5,-1,3,1) + (-5)(-1,-5,-1,-9,2) + (1)(5,25,0,31,0)$.
(d)$\{(-2,1,0,0,0,0,0),(1,0,-3,1,0,0,0),(-2,0,1,0,5,-1,1)\}$
$dim(\text{Nul}(A)) = 3$.
(e)$\{(1,2,0,-1,0,0,2),(0,0,1,3,0,0,-1),(0,0,0,0,1,0,-5),(0,0,0,0,0,1,1)\}$
(f)$(1,2,1,2,-1,5,11) = 1(1,2,0,-1,0,0,2)+1(0,0,1,3,0,0,-1)+(-1)(0,0,0,0,1,0,-5)+5(0,0,0,0,0,1,1)$
(g)$dim(\text{Nul}(A^T)) = 1$.

[10] 2. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 9 \\ 7 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ k \\ -3 \\ 2k \end{bmatrix}$.

(a) Find a condition on x_1, x_2, x_3 and x_4 that is necessary and sufficient for the vector $\mathbf{v}$ to be in the subspace Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

(b) Are the following sets of vectors linearly dependent or independent?

i. $\{\mathbf{u}_1, \mathbf{u}_3\}$

ii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\}$
iii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

(c) For what values of k is $\mathbf{w}$ in the span of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$?

(d) Give a basis for Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ such that none of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is included in your basis.

Answers:
2.
(a) $3x_1 + x_2 - 6x_3 + 8x_4 = 0$
(b)
(i) LI
(ii) LI
(iii) LD
(c) $k = -21/17$.
(d) $\{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3\}$

[10] 3. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ and define the linear transformations

$T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ by $T_1(\mathbf{x}) = A\mathbf{x}$, and

$T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ by $T_2(\mathbf{x}) = A^T\mathbf{x}$.

Also, let $\mathcal{S}$ denote the unit square in $\mathbb{R}^2$, that is

$$\mathcal{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1 \right\},$$

and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ defined by

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

(a) What is $T_1\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$? What is $T_2\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$?

(b) Find $(T_2 \circ T_1)(\mathcal{S})$. Draw pictures of $\mathcal{S}$ and $(T_2 \circ T_1)(\mathcal{S})$.
($(T_2 \circ T_1)(\mathcal{S})$ denotes the set of images of the vectors in the unit square $\mathcal{S}$, under the linear transformation $T_2 \circ T_1$.)

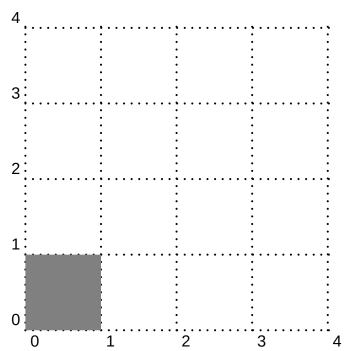
(c) Find $(T_1 \circ T_2)(\mathcal{L})$.

(d) Fill in the following table with YES or NO as appropriate.

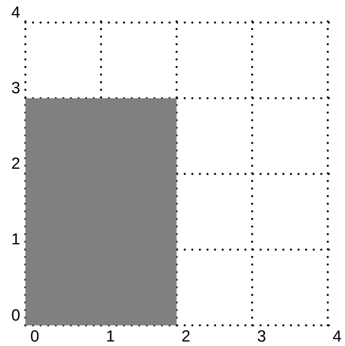
	onto	one-to-one
T_1		
T_2		
$T_1 \circ T_2$		
$T_2 \circ T_1$		

Answers:
3.
(a) $T_1\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$
$T_2\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ 3. (b)

$\mathcal{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \right\}$



$(T_2 \circ T_1)(\mathcal{S}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 0 \leq x_1 \leq 2 \text{ and } 0 \leq x_2 \leq 3 \right\}$



(c) $(T_1 \circ T_2)(\mathcal{L}) = \{(6, 2, 4)\}$.
(d)

	onto	one-to-one
T_1	no	yes
T_2	yes	no
$T_1 \circ T_2$	no	no
$T_2 \circ T_1$	yes	yes

[10] 4. Let

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 0 & -4 \end{bmatrix},\ B = \begin{bmatrix} 6 & 0 \\ -2 & 8 \\ 1 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 1 \\ -3 & 3 \end{bmatrix}$$

(a) i. Evaluate $AB + 3C$.
ii. If possible, find a matrix X such that $3CX = I - ABX$.
(You should try to solve for X using matrix algebra.)
iii. What is the rank of the 5×5 matrix

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}?$$

(b) Let Y be an $n \times 2$ matrix. Fill in the blanks with *must*, *might* or *cannot* to make each of the following statements true.

i. If Y has two pivot positions then YC __________ be invertible and CY^T __________ be invertible.
ii. If Y has one pivot position then YC __________ have linearly independent columns, and CY^T __________ have linearly independent columns.

Answers:
4.
(a)
(i)

$$AB + 3C = \begin{bmatrix} 24 & 12 \\ -7 & 13 \end{bmatrix}$$

(ii)

$$X = \begin{bmatrix} 13/396 & -1/33 \\ -7/396 & 2/33 \end{bmatrix}$$

(iii) 4
(b)
(i) If Y has two pivot positions then YC *might* be invertible and CY^T *might* be invertible.
(ii)If Y has one pivot position then YC *cannot* have linearly independent columns, and CY^T *might* have linearly independent columns.

[6] 5. Let $A = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -2 & 0 & -3 \\ 1 & -2 & 3 & 4 \end{bmatrix}$

(a) Find the following determinants:

i. $\texttt{det}\,(A)$
ii. $\texttt{det}\,(-3A)$
iii. $\texttt{det}\,(A^{-2})$
iv. $\texttt{det}\,(PAP^{-1})$ where P is a 4×4 invertible matrix.
v. $\texttt{det}\,(BAB)$ where B is a singular (i.e. non-invertible) matrix.
vi. $\texttt{det}\,(D)$ where D is the reduced row echelon form of the matrix A.

(b) Use the determinant of A^{-1} to find $\texttt{adj}\,(A^{-1})$.

[4] 6. Find all values of s for which the following system is inconsistent.
For full marks show the work that justifies your answer.

$$3sx_1 + 2x_2 = 4$$
$$6x_1 + sx_2 = -4$$

Answers:
5.
(a)
(i) -57
(ii) -4617
(iii) $1/3249$
(iv) -57
(v) 0

(vi) 1

(b) $(-1/57)\begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -2 & 0 & -3 \\ 1 & -2 & 3 & 4 \end{bmatrix}$

6. The augmented matrix of this system row reduces to $\begin{bmatrix} 6 & s & -4 \\ 0 & 2-(1/2)s^2 & 4+4s \end{bmatrix}$.

The system will be inconsistent if and only if there is a pivot position in the last column. Since the $(1,1)$ position is a pivot position, the system is inconsistent if and only if the $(3,2)$ position is a pivot position. Thus inconsistency of the system is equivalent to $2-(1/2)s^2 = 0$ and $s \neq -2$. Therefore the system is inconsistent if $s = 2$ and consistent if $s \neq 2$.

[10] 7. An $n \times n$ matrix B is called idempotent if $B^2 = B$.

(a) Suppose that B is an $n \times n$ idempotent matrix

i. Show that $\texttt{det}\, B = 0$ or $\texttt{det}\, B = 1$.

ii. Show that if $\texttt{det}\, B = 1$ then $B = I$. (I is the $n \times n$ identity matrix.)

iii. Show that $I - B$ is also idempotent.

(b) For what values of a and b is $\begin{bmatrix} 2 & 3 \\ a & b \end{bmatrix}$ idempotent?

(c) Let A be any $n \times n$ matrix. Show that

$$\begin{bmatrix} A & \frac{1}{k}A \\ k(I-A) & I-A \end{bmatrix}$$

is idempotent, where k is any non-zero scalar.

7.Answers

(a) Let $x = \texttt{det}\, B$.

(i) $x^2 = (\texttt{det}\, B)^2 = \texttt{det}\,(B^2) = \texttt{det}\, B = x$. therefore $x^2 - x = 0$ i.e. $x(x-1) = 0$ and so $x = 0$ or $x = 1$.

(ii) Since $\texttt{det}\, B \neq 0$, B^{-1} exists and so

$B = IB = (B^{-1}B)B = B^{-1}(BB) = B^{-1}B = I$.

(iii) $(I-B)^2 = (I-B)(I-B) = I - B - B + B^2 = I - B - B + B = I - B$.

(b) $a = -2/3$ and $b = -1$.

(c)

$$\begin{bmatrix} A & \frac{1}{k}A \\ k(I-A) & I-A \end{bmatrix}\begin{bmatrix} A & \frac{1}{k}A \\ k(I-A) & I-A \end{bmatrix}$$

$$= \begin{bmatrix} AA + (\frac{1}{k}A)(k(I-A)) & A(\frac{1}{k}A) + (\frac{1}{k}A)(I-A) \\ k(I-A)A + (I-A)k(I-A) & k(I-A)(\frac{1}{k}A) + (I-A)(I-A) \end{bmatrix}$$

$$= \begin{bmatrix} A^2 + A - A^2 & \frac{1}{k}(A^2 + A - A^2) \\ k(A - A^2 + I - 2A + A^2) & A - A^2 + I - 2A + A^2 \end{bmatrix}$$

$$= \begin{bmatrix} A & \frac{1}{k}A \\ k(I-A) & I-A \end{bmatrix}$$

[6] 8. Let V be the subspace of the space of all 2×2 matrices defined by

(a) Is O (the 2×2 zero matrix) in V?

(b) Is I_2 (the 2×2 identity matrix) in V?

(c) For what a is $\begin{bmatrix} 2 & 2 \\ 3 & a \end{bmatrix}$ in V?

(d) Find a basis for V.

(e) Write the matrix you found in part (c) as a linear combination of the basis matrices you found in part (d).

[4] 9. Which of the following sets are subspaces of P_2, the space of polynomials of degree at most 2. If a set is a subspace, give a basis of the subspace. If a set is not a subspace, explain why it is not a subspace.(No marks unless you give an adequate explanation of why a set is not a subspace.)

(a) $\{p(x) : p'(1) = 0\}$

(b) $\{p(x) : \int_0^1 p(x)dx = 1\}$

Answers:
8.
(a) Yes
(b) No
(c) $a = -3$
(d)

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

(e)

$$\begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

9.
(a) $\{x^2 - 2x, 1\}$
(b) The zero polynomial is not in the set and so it is not a subspace.

[10] 10. Given the points $P(0,0,1)$, $Q(1,1,2)$, $R(4,6,5)$ and $S(6,11,10)$, find the following:

(a) a normal to the plane containing the points P, Q and R.

(b) the standard equation of the plane containing the points P, Q and R. (The standard equation has the form $ax + by + cz = d$.)

(c) the standard equation of the plane through the origin parallel to the plane found in part (b).

(d) the area of triangle PQR

(e) the volume of the parallelepiped three of whose sides are PQ, PR and PS.

(f) the distance between the point S and the plane found in part (b).

Answers:
10.
(a) $(1, 0, -1)$
(b) $x - z = -1$
(c) $x - z = 0$
(d) $\sqrt{2}$
(e) 6
(f) $(3/2)\sqrt{2}$

[10] 11. The identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

is true for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in R^3.

(a) Fill in the blanks with *must*, *might* or *cannot* to make each of the following statements true.

i. The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ __________ lie in the span of the vectors $3\mathbf{v}$ and $5\mathbf{w}$
ii. The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ __________ be orthogonal to the vector $2\mathbf{v} \times (-4\mathbf{w})$
iii. The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ __________ be a solution of $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{x} = 0$.
iv. The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ __________ be parallel to the vector $\mathbf{u}$.

(b) Give a specific numeric example of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}$.

(c) Use the identity to simplify $(\mathbf{u} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w})$.

(d) Apply the identity to write $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Answers:
11.
(a)(i) The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *must* lie in the span of the vectors $3\mathbf{v}$ and $5\mathbf{w}$

(ii) The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *must* be orthogonal to the vector $2\mathbf{v} \times (-4\mathbf{w})$
(iii) The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *might* be a solution of $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{x} = 0$.
(iv) The vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *cannot* be parallel to the vector $\mathbf{u}$.
(b)$\mathbf{u} = (0, 0, 0)$, $\mathbf{v} = (1, 0, 0)$, $\mathbf{w} = (0, 1, 0)$
(c)$(\mathbf{u} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}))\mathbf{w}$
(d) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (-(\mathbf{w} \cdot \mathbf{v}))\mathbf{u} + (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$

[7] 12. Consider the planes $4x + y - 3z = 7$ and $2x - 3y + 3z = 4$.

(a) Find their line of intersection.

(b) For each of the above two planes, find a normal, and the find the angle between these two normals (*in radians, 2 decimal places*).

[3] 13. Find the point of intersection of the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

with the plane containing both the y-axis and the z-axis.

Answers:
12.
(a)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25/14 \\ -1/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 9 \\ 7 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b) $\mathbf{n_1} = (4, 1, -3)$ and $\mathbf{n_2} = (2, -3, 3)$
$\cos(\mathbf{n_1}, \mathbf{n_2}) = -\frac{2}{\sqrt{143}}$
13.$(0, -14/5, 6/5)$.