1. Approximate the given definite integral correct to 5 decimal places (i.e., within $\pm 0.5 \times 10^{-5}$):
$\int_0^1 \frac{1-\cos x}{x^2} dx$

2. A function $f(x)$ has Maclaurin series: $1 + x^2 + \frac{x^4}{4} + \frac{x^6}{9} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}$

find $f^{(k)}(0)$ for all positive integer k.

3. (a) Use the binomial series to find the Maclaurin series for $f(x) = \arcsin x$ and its radius of convergence (Hint: $\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$).
(b) Use the series in Part (a) to evaluate the following limit:

$$\lim_{x \to 0} \frac{x - \arcsin x}{x^3}$$

4. (a) Find the third degree Taylor polynomial $T_3(x)$ for the function $f(x) = \ln(1+2x)$ centered at $a = 1$.
(b) Use Taylor's inequality (or Lagrange's remainder) to estimate the error in using $T_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

5. Given the curve $\mathcal{C}$ having parametric equations: $\begin{cases} x = 9 - t^2 \\ y = t^3 - 16t \end{cases}$ where $t \in \mathcal{R}$

   i) Find dy/dx and d^2y/dx^2.
   ii) Find the x and y intercepts and coordinates of the points on $\mathcal{C}$ where the tangent line is vertical or horizontal.
   iii) Sketch the graph of $\mathcal{C}$ showing the orientation of the curve.
   iv) The curve forms a loop. **Set up, but do not evaluate,** the integrals needed to find the area enclosed by the loop and the length of the loop.

6. Given the polar curves $r = 1 + 2\sin\theta$ and $r = 2$, do the following:

   i) Sketch both graphs on the same axes.
   ii) Find all the points of intersection for $\theta \in [0, 2\pi]$.
   iii) Find the area of the region outside the circle and inside the limaçon.
   iv) **Set up, but do not evaluate,** the integral needed to find the length of the inner loop of the limaçon.

7. Sketch and give the name of the following surfaces:
   i) $x^2 + y^2 = z^2 + 9$
   ii) $z^2 = 9 - 4x^2 - y^2$

8. Let $\mathbf{r}(t) = \langle \sin t, \sqrt{2}\cos t, \sin t \rangle$.

   i) Compute the velocity, speed and acceleration.
   ii) Find the curvature and the tangential and normal components of acceleration.
   iii) Find an equation of the quadric surface on which this space curve lies. Sketch $\mathbf{r}(t)$ for $0 \le t \le 2\pi$. (You might want to sketch the curve on the surface, to help you make a good graph.)

9. Find the limit if it exists or show that it does not exist.
(a) $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$ (b) $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$

10. Find the equations of the tangent plane and normal line of the surface $x^2 + y^2 + z = 6$ at the point $P(2, 1, 1)$.

11. If $z = f(x, y)$ is implicitly defined by $x^2z + \sin(yz) = y\sec z$, find $\frac{\partial z}{\partial x}$.

12. If $z = f(x^2 - y^2, 2xy)$ find $\dfrac{\partial^2 z}{\partial x^2}$. Assume that second order partial derivatives of f are continuous.

13. Find and classify the critical points of $f(x, y) = 6xy^2 - 2x^3 - 3y^4$.

14. Use the method of Lagrange multipliers to find the smallest and largest values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

15. Let $z = f(x, y)$ be a surface and $f(x, y) = c$ a level curve on that surface. Show that the gradient of f is always perpendicular to the level curve (Hint: You may want to represent the level curve by the vector equation $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, and then use the chain rule.).

16. **Set up, but do not evaluate**, the integral needed to find the volume of the region E lying inside $x^2 + 4y^2 = 4$, above the xy-plane and below $z = 2 - x$. Sketch the region.

Evaluate the multiple integrals in problems 17–20:

17. $\displaystyle\int_0^9 \int_{\sqrt{x}}^3 xy \sin y^6 dydx$

18. $\displaystyle\iint_R (x - y)^5(x + y)^3 dA$ where R is the triangular region bounded by the coordinate axes and the line $x + y = 1$ (Hint: Use an appropriate change of variables).

19. $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dzdydx$

20. $\displaystyle\iiint_S (x^2 + y^2 + z^2)^2 dV$ where S is the region that lies above the cone $z = \sqrt{\dfrac{x^2 + y^2}{3}}$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

ANSWERS

1. $I \simeq \frac{1}{2!} - \frac{1}{3(4!)} + \frac{1}{5(6!)} \simeq 0.48639$

   with $|\text{error}| \leq \frac{1}{7(8!)} \simeq 0.35 \times 10^{-5}$

2. If k is odd then $f^{(k)}(0) = 0$; if $k = 0$ then $f^{(0)}(0) = 1$; if $k > 0$ is even then $f^{(k)}(0) = \frac{4k!}{k^2}$

3. $\arcsin x = x + \displaystyle\sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$ with $R = 1$;

   The limit is equal to $-\frac{1}{6}$

4. $T_3(x) = \ln 3 + \frac{2(x-1)}{3} - \frac{2(x-1)^2}{9} + \frac{8(x-1)^3}{81}$

   $R_3(x) = \frac{-96(x-1)^4}{(1+2z)^4 4!}$ where z is between 1 and x.

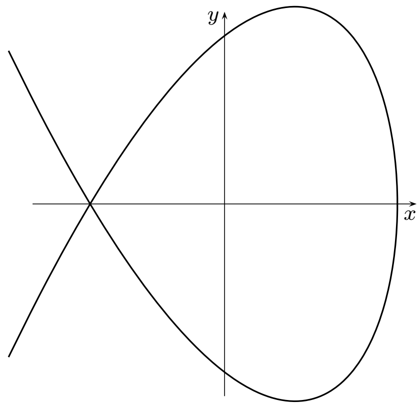
   $|R_3(x)| \leq \frac{96}{4!(16)^2} = \frac{1}{64}$ since $0.5 < z < 1.5$

5. $\frac{dy}{dx} = \frac{16-3t^2}{2t}$ $\quad \frac{d^2y}{dx^2} = \frac{16+3t^2}{4t^3}$

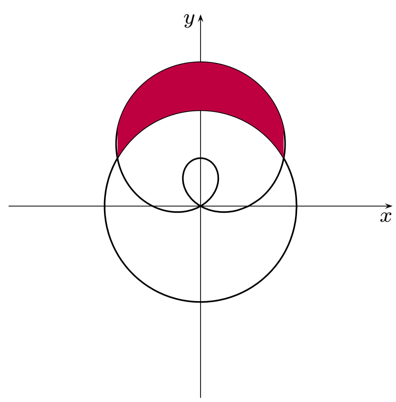
   intercepts are $(-7, 0)$,$(9, 0)$, $(0, -21)$ and $(0, 21)$

   H.T. at $(\frac{11}{3}, \pm\frac{128}{3\sqrt{3}})$ and V.T. at $(9, 0)$

$\mathcal{A} = 2\int_{-4}^{0}(t^3 - 16t)(-2t)dt = 4\int_0^4(t^4 - 16t^2)dt$
$\mathcal{L} = \int_{-4}^{4}\sqrt{4t^2 + (3t^2 - 16)^2}\,dt$



6. The points of intersection are $(2, \frac{\pi}{6})$ and $(2, \frac{5\pi}{6})$;
   $\mathcal{A} = \frac{5\sqrt{3}}{2} - \frac{\pi}{3}$ and $\mathcal{L} = \int_{7\pi/6}^{11\pi/6}\sqrt{5 + 4\sin\theta}\,d\theta$ or equivalent.



7. (i) Hyperboloid of one sheet; (ii) Ellipsoid
8. $\mathbf{v}(t) = \langle \cos t, -\sqrt{2}\sin t, \cos t\rangle$
   $\mathbf{a}(t) = \langle -\sin t, -\sqrt{2}\cos t, -\sin t\rangle$
   speed=$\frac{ds}{dt} = \sqrt{2}$ therefore $a_T = \frac{d^2s}{dt^2} = 0$
   $\kappa = \frac{1}{\sqrt{2}}$ which implies that $a_N = \kappa(\frac{ds}{dt})^2 = \sqrt{2}$
   The curve is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 2$ and the plane $x = z$.
9. (a) Note that $f(x, x) = 0$ while $f(0, y) = -1$ for $x \neq 0$ and $y \neq 0$; these lead to two different limits as $(x, y) \to (0, 0)$ so the limit does not exist.
   (b) Note that $0 \leq \left|\dfrac{3x^2y}{x^2 + y^2}\right| \leq 3|y|$ so the limit is zero by Squeeze Theorem.
10. $4x + 2y + z = 11$ and $\langle x, y, z\rangle = \langle 2, 1, 1\rangle + t\langle 4, 2, 1\rangle$ where $t \in \mathcal{R}$
11. $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xz}{x^2 + y\cos(yz) - y\sec z\tan z}$
12. Let $u = x^2 - y^2$ and $v = 2xy$ it follows that $\dfrac{\partial^2 z}{\partial x^2} = 2(f_u + 2x^2 f_{uu} + 2y^2 f_{vv} + 4xy f_{uv})$
13. The critical points are $(0, 0)$, for which the second derivative test is inconclusive, $(1, -1)$ and $(1, 1)$, which are both local maxima.

14. The largest value of $f(x, y)$ is $\frac{1}{2}$ occurring at $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

    The smallest value of $f(x, y)$ is $-\frac{1}{2}$ occurring at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

15. Assume that f and $\mathbf{r}$ are differentiable. Note $\frac{df}{dt} = 0$ since $f(x, y) = c$.

    Also by chain rule, $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$; this implies that $\nabla f(x, y) \cdot \mathbf{r}'(t) = 0$

16. $V(E) = \displaystyle\int_{-1}^{1} \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \int_{0}^{2-x} dz dx dy$

17. $I = \displaystyle\int_{0}^{3} \int_{0}^{y^2} xy \sin y^6 dx dy = \frac{1 - \cos(729)}{12}$

18. Let $u = x - y$ and $v = x + y$ then $I = \displaystyle\int_{0}^{1} \int_{-v}^{v} u^5 v^3 \left(\frac{1}{2}\right) du dv = 0$

19. $\displaystyle\int_{0}^{2\pi} \int_{0}^{2} \int_{r}^{2} r^3 dz dr d\theta = \frac{16\pi}{5}$

20. $\displaystyle\int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{0}^{2} \rho^6 \sin\phi \, d\rho d\phi d\theta = \frac{128\pi}{7}$