1. (2 points) How many three-letter words (these words need not have meanings) can be made from six letters "fghijk" if

   a) repetition of letters is not allowed? b) repetition of letters is allowed?

2. (2 points) Evaluate the expressions: a) ${}_8C_4$ b) ${}_{10}P_7$

3. (5 points) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$. Find the following:
   a) $A \cap B$ b) $A \cup B$ c) $A - B$ d) $B - A$ e) Number of subsets of A

4. (2 points) Draw a Venn diagram and show by hatching the following set: $\overline{A} \cap \overline{(B \cap C)}$

5. (2 points) Draw a Venn diagram to represent the relationship between A, B and C:
   $(A \cap B) \subset C$, $A \not\subseteq C$ and $B \not\subseteq C$

6. (6 points) For each expression, name the property from the given list and say whether it is a set property, a network property, or a logic property: *associativity, commutativity, distributivity, identity, idempotent, de Morgan, closure, complement, property of 1 (or 0), tautology, contradiction.*

   a) $A + B = B + A$ ______________________________
   b) $A \cup \overline{A} = U$ ______________________________
   c) $\overline{A + B} = \overline{A} \cdot \overline{B}$ ______________________________
   d) $A \cup (B \cup C) = (A \cup B) \cup C$ ______________________________
   e) $(p \vee c) \leftrightarrow p$ ______________________________
   f) $[p \vee (q \wedge r)] = [(p \vee q) \wedge (p \vee r)]$ ______________________________

7. (3 points) Find a truth table for the logical expression: $(p \vee q)\overline{\vee}(p \rightarrow r)$

8. (3 points) Use a truth table to determine whether or not $p \rightarrow q$ is equivalent to $\sim (p \wedge \sim q)$

9. (6 points) Use a truth table to determine whether the argument is valid or not.

   H: Rhombus R is a square or a parallelogram.
   Rhombus R is a parallelogram.

   C: Rhombus R is not a square.

10. (3 points) Use a Venn diagram to determine the validity of the argument.

    H: Some seals swim.
    All animals that swim have flippers.

    C: Some seals have flippers.

11. (4 points) If a number ends in five or zero then the number is divisible by five.

    a) Write the converse, the contrapositive, and the inverse of the implication above.
    b) Say which among the four statements are equivalent.

12. (2 points) Draw a network to represent the given Boolean expression: $AB(A\,\overline{C} + D)E + CD$

13. (3 points) Find a Boolean table for the expression: $(A+B)(A+C)+\overline{B}C$

14. (5 points) Simplify each expression, justifying each step using properties of Boolean algebra.

a) $A(B+C)+A\,\overline{B}$

b) $ABC+\overline{A}\,C+\overline{B}$

15. (3 points) Classify each system below (*without solving*) as dependent or independent, consistent with a unique solution, consistent with infinitely many solutions or inconsistent. Justify your answers.

a) $\begin{aligned} 2x+3y &= 3 \\ -4x-6y &= 11 \end{aligned}$ (1)

b) $\begin{aligned} 7x+y &= 4 \\ 4x+y &= 7 \end{aligned}$ (1)

c) $\begin{aligned} 2x+3y &= 1 \\ -8x-12y &= -4 \end{aligned}$ (1)

16. (6 points) Given the system $\begin{cases} 3x+5y = 0 \\ x-2y = 11 \end{cases}$

a) Estimate the solution by graphing. **b)** Solve the system by substitution. **c)** Solve the system by multiplication-addition to verify your answer in (b).

17. (12 points) Let $A=\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ $B=\begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & -2 \end{bmatrix}$ $C=\begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ $D=\begin{bmatrix} 6 & 2 & -3 \\ 3 & 1 & -2 \\ 1 & 0 & 4 \end{bmatrix}$

$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find each of the following, if possible. If an operation is not possible, say why.

a) $5D-3C$ **b)** $6D^T$ **c)** B^TA **d)** $(I-A)^T$

e) BA **f)** DC **g)** $B+3I$

18. (9 points) Given $A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ find A^{-1} using elementary row operations and verify your answer.

19. (3 points) Given $A=\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, explain why A^{-1} does not exist.

20. (3 points) Given $A=\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$ find A^{-1}.

21. (7 points) Solve the following system by Gaussian or Gauss-Jordan Elimination, if possible.

$$\begin{aligned} 2x+y-2z &= 10 \\ x+y+4z &= -9 \\ 5x+4y+3z &= 4 \end{aligned}$$

22. (4 points) Solve the following system by Gaussian or Gauss-Jordan Elimination, if possible.

$$\begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned}$$

23. (2 points) Suppose that the augmented matrix of a linear system is reduced to the following form. For what values of k does the system have **a)** a unique solution? **b)** no solution?

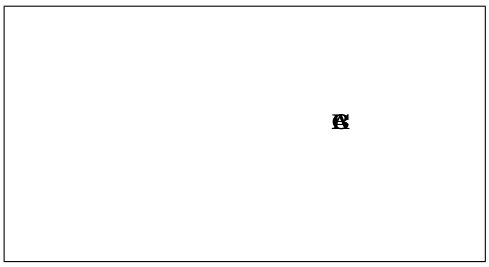
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & k & 5 \end{array}\right]$$

24. (3 points) Prove by mathematical induction that for all positive integers n

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

ANSWERS

1. (a) ${}_6P_3 = 120$ (b) $6^3 = 216$
2. (a) 70 (b) 604800
3. (a) $\{2, 4\}$, (b) $\{1, 2, 3, 4, 6, 8\}$, (c) $\{1, 3\}$, (d) $\{6, 8\}$, (e) 16
4. This is equal to $\overline{A \cup (B \cap C)}$, so all the area outside A and outside $B \cap C$.
5. the diagram is as follows:



6. (a) commutative, network
   (b) complement, set
   (c) de Morgan, network
   (d) associative, set
   (e) identity, logic
   (f) distributive, logic
7. This expression is a contradiction (always false).
8. show that $(p \to q) \leftrightarrow \sim (p \wedge \sim q)$ is a tautology.
9. The argument is invalid; show $[(p \vee q) \wedge q] \to \sim p$ is not a tautology.
10. The argument is valid; $S \cap N \neq \emptyset$ and $N \subseteq F$ implies that $S \cap F \neq \emptyset$.

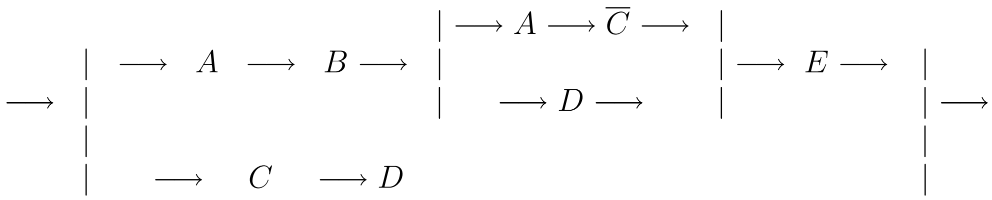
11. Converse: If a number is divisible by five then the number ends in five **or** in zero.

    Inverse: If a number does not end in five **and** and does not end in zero then the number is not divisible by five.

    Contrapositive: If a number is not divisible by five then the number does not end in five **and** does not end in zero.

    converse is equivalent to inverse and original is equivalent to contrapositive.

12. The network is as follows:



13. It is 0 when $A = C = 0$, $B = 1$ and when $A = B = C = 0$; it is 1 in all the other circumstances.

14. (a) A (b) $\overline{B} + C$

15. (a) parallel lines, independent and inconsistent, no solution

    (b) different slopes, independent and consistent, a unique solution

    (c) same line, dependent and consistent, infinitely many solutions.

16. (a) The graph suggests that $x \simeq 5$ and $y \simeq -3$, which turns out to be the exact answer by (b) or (c).

y

17. (a) $\begin{bmatrix} 18 & 4 & -18 \\ 9 & -7 & -10 \\ -1 & -3 & 14 \end{bmatrix}$ (b) $\begin{bmatrix} 36 & 18 & 6 \\ 12 & 6 & 0 \\ -18 & -12 & 24 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 8 \\ 6 & 0 \\ 7 & -4 \end{bmatrix}$

    (d) $\begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}$ (e) impossible (f) $\begin{bmatrix} 22 & 17 & 0 \\ 10 & 8 & -1 \\ 12 & 6 & 9 \end{bmatrix}$ (g)impossible.

18. $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

19. A can not be reduced to I (a row of zeros appears in the left hand side of the $[A|I]$ matrix.

20. $A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

21. $x = 1$, $y = 2$, $z = -3$

22. inconsistent

23. If $k \neq 0$, the system has a unique solution; if $k = 0$, the system has no solution.

24. P_1 is true as $1(2) = \frac{1}{3}(1)(2)(3)$.
Assume P_k is true then $1(2) + 2(3) + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$ which implies (add $(k+1)(k+2)$ to both sides)

$$\begin{aligned} 1(2) + 2(3) + \cdots + k(k+1) + (k+1)(k+2) &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

This shows that P_{k+1} is true. Therefore, P_n is true for all positive integers n.