(3) 1. a) Five people are taking part in a spelling competition. In how many ways can, first, second and third place be awarded?

b) Bill has a choice of 4 pants and 6 shirts. In how many different ways can he dress?

c) What is the number of 3-element subsets of a set with 7 elements?

(3) 2. Let $A = \{1, 3, 5\}$.

a) List all proper subsets of A.

b) If $U = \{x \in N : 1 \leq x \leq 10\}$, what is $\overline{A}$?

c) Suppose B is any subset of U such that $A \subseteq B$. What can you say about $A \cap B$? What can you say about $A \cup B$?

(4) 3. Draw Venn diagrams and show by hatching the following sets:

a) $(A \cup C) \cap \overline{B}$
b) $(A - C) \cup \overline{(B \cap C)}$

(6) 4. For each expression, name the property from the given list and say whether it is a set property, a network property, or a logic property: *associativity, commutativity, distributivity, identity, idempotent, de Morgan, closure, complement, property of 1 (or 0), tautology, contradiction.*

a) $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ ____________________

b) $A + 1 = 1$ ____________________

c) $A \cdot (B + C) = A \cdot B + A \cdot C$ ____________________

d) $A \cup B \subseteq U$ ____________________

e) $A \cup A = A$ ____________________

f) $(p \vee t) \leftrightarrow t$ ____________________

(3) 5. Use a truth table to determine whether or not $p \wedge (q \vee r)$ is equivalent to $p \vee (q \wedge r)$

(3) 6. Determine whether the given expression is a tautology, contradiction or neither?

$(p \rightarrow \sim q) \;\overline{\wedge}\; (p \veebar q)$

(5) 7. Use a truth table to determine whether the argument is valid or not.

H: If it snows, then it is winter.
It is winter and it snows.

C: It is winter if and only if it snows.

(3) 8. Use a Venn diagram to determine the validity of the argument.

H: Some friends lie.
Some friends are nice.

C: Friends who are nice do not lie.

(4) 9. If it is not 7 p.m. then I am not late for my bus.

a) Write the converse, the contrapositive, and the inverse of the implication above.
b) Say which among the four statements are equivalent.

(5) 10. Find a Boolean table for each expression:

a) $(A+B)\overline{B}$

b) $\overline{(A+B)}+A\overline{C}$

(2) 11. Draw a network to represent the given boolean expression:
$B(\overline{A}B+CD)+A\overline{C}$

(5) 12. Simplify each expression, justifying each step using properties of Boolean algebra.

a) $AB+A\overline{B}+B$ (2)

b) $A+B+C+\overline{A}\,\overline{C}$ (3)

(6) 13. Classify each system below (*Without solving*) as dependent or independent, consistent with one solution only, consistent with infinitely solutions or inconsistent. Justify your answers.

a) $\begin{aligned} 3x- \quad 2y &= 3 \\ y &= 2x+1 \end{aligned}$ (2)

b) $\begin{aligned} 2x-3y &= 3 \\ -4x+6y &= -6 \end{aligned}$ (2)

c) $\begin{aligned} 4x-2y &= 5 \\ 2x-y &= -2 \end{aligned}$ (2)

(4) 14. Given the system

$$\begin{aligned} 3x-y &= 7 \\ x+3y &= 9 \end{aligned}$$

a) Estimate the solution by graphing.

b) Solve the following system algebraically by substitution or elimination.

(12) 15. Given: $A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}$ $C=\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ $D=\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$

$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find each of the following, if possible. If an operation is not possible, say why.

a) $3D-2C$ (2)

b) CC^T (2)

c) B^{-1} (2)

d) $D+I$ (1)

e) IC (1)

f) AD (3)

g) DI (1)

(3) 16. Given $A=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -3 \\ 0 & 2 & -11 \end{bmatrix}$, explain why A^{-1} does not exist.

(8) 17. Given $A=\begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$ find A^{-1} using elementary row operations.

(4) 18. Given the linear system

$$\begin{aligned} x+3y+z &= 4 \\ 2x+2y+z &= -1 \\ 2x+3y+z &= 3 \end{aligned}$$

a) write the system in matrix form $AX = B$. (2)

b) If $A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$, solve the system using A^{-1}. (3)

(3) 19. Suppose that the augmented matrix of a linear system is reduced to the following form. Does this system have a solution? If so find the solution set. If not explain why.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(7) 20. Solve the following system by Gaussian or Gauss-Jordan Elimination, if possible.

$$\begin{aligned} -x-2y+3z &= 1 \\ x+y+2z &= 8 \\ 3x-7y+4z &= 10 \end{aligned}$$

(4) 21. Solve the following system by Gaussian or Gauss-Jordan Elimination, if possible.

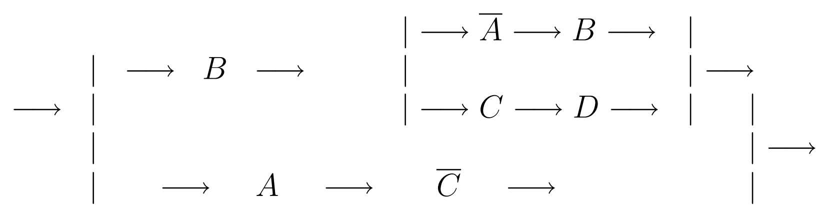
$$\begin{aligned} x-y+z &= 5 \\ 3x+2y-z &= 3 \\ x+4y-3z &= 9 \end{aligned}$$

(3) 22. Prove by mathematical induction that for all positive integers n

$$1+3+5+\cdots+(2n-1) = n^2$$

ANSWERS

1. (a) ${}_5P_3 = \frac{5!}{2!} = 60$
   (b) $4 \times 6 = 24$
   (c) ${}_7C_3 = \frac{7!}{3!4!} = 35$
2. (a) $\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}$
   (b) $\overline{A} = \{2,4,6,7,8,9,10\}$
   (c) $A \cap B = A$ and $A \cup B = B$
3. (a) The regions that are both in A and C but not in B
   (b) All the regions excluding $B \cap C$
4. (a) de Morgan, logic
   (b) property of 1, network
   (c) distributive, network
   (d) closure, set
   (e) idempotent, set
   (f) tautology, logic

5. No, they are not equivalent
6. It is neither (it is equivalent to $p \leftrightarrow q$)
7. It is a valid argument (Show $(p \to q) \wedge (q \wedge p) \to (p \leftrightarrow q)$ is a tautology)
8. Invalid, can have $(F \cap N) \cap L \neq \emptyset$
9. Converse: If I am not late for my bus then it is not 7 p.m.
   Contrapositive: If I am late for my bus then it is 7 p.m.
   Inverse: If it is 7 p.m. then I am late for my bus.
   The original is equivalent to contrapositive and the converse is equivalent to inverse.
10. (a) It is 1 when $A = 1$ and $B = 0$ and 0 otherwise.
    (b) It is 1 when $A = B = 1$, $C = 0$ or when $A = 1$, $B = C = 0$ or when $A = B = 0$ and $C = 1$ or when $A = B = C = 0$; it is 0 in all four other situations.
11. 

12. (a) It simplifies to $A + B$
    (b) It simplifies to 1.
13. (a) Independent, consistent (a unique solution)
    (b) dependent, consistent (infinitely many solutions)
    (c) independent, inconsistent (no solution)
14. Answer is $(3, 2)$
15. (a) $\begin{bmatrix} 1 & 37 & 5 \\ 12 & 4 & 20 \end{bmatrix}$
    (b) $\begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$ (c) $\begin{bmatrix} 4/3 & 5/3 \\ 1/3 & 2/3 \end{bmatrix}$
    (d) $D + I$ is not possible; (e) $IC = C$
    (f) $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$ (g) DI is not possible.
16. A can not be reduced to I (a row of zeros appears in the left hand side of the $[A|I]$ matrix.
17. $A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$
18. $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$
    $X = A^{-1}B = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$

19. $x = 10$, $y = -3$, $z = -1$

20. $x = 3$, $y = 1$, $z = 2$

21. inconsistent

22. P_1 is true as $1 = 1^2$.
Assume P_k is true then $1+3+5+\cdots+(2k-1) = k^2$ which implies (add $2k+1$ to both sides)

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

This shows that P_{k+1} is true.