# 1 Continuity

1. Use the definition of continuity to show that the function

$$f(x) = \begin{cases} 2x & \text{for } x \leq 2 \\ 3 & \text{for } x > 2 \end{cases}$$

is not continuous at $x = 2$.

2. The function $f(x)$ is defined by

$$f(x) = \begin{cases} 2x & \text{for } x \leq 1 \\ 3 & \text{for } x > 1 \end{cases}$$

   (a) Sketch the graph of $f(x)$.

   (b) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

   (c) Is $f(x)$ continuous at $x = 1$? Explain.

3. None of the following is continuous at $x = 1$. For each, state the condition for continuity that is not satisfied:

   (a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

   (b) $f(x) = \begin{cases} 2x & \text{for } x < 1 \\ x^2 & \text{for } x \geq 1 \end{cases}$

   (c) $f(x) = \begin{cases} \sqrt{x} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$

4. Using the definition of continuity, prove that the following function is continuous at $x = -2$:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2} & \text{if } x \neq -2 \\ -5 & \text{if } x = -2 \end{cases}$$

5. Determine a value for $b$ which makes $g(x)$ continuous.

$$g(x) = \begin{cases} bx & \text{if } x < 3 \\ 6 & \text{if } x \geq 3 \end{cases}$$

6. If

$$f(x) = \begin{cases} 3x + 1 & \text{when } x < 2 \\ 1 & \text{when } x = 2 \\ 5 - x & \text{when } x > 2 \end{cases}$$

   (a) Sketch the graph of $f(x)$.

   (b) Find (i) $\lim_{x \to 2^-} f(x)$ and (ii) $\lim_{x \to 2^+} f(x)$

   (c) Is $f(x)$ continuous at $x = 2$? Explain.

7. If

$$f(x) = \begin{cases} 3 + x & \text{if} & x \leq 1 \\ 3 - x & \text{if} & x > 1 \end{cases}$$

   (a) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-}$.

   (b) Does $\lim_{x \to 1} f(x)$ exist? Explain.

   (c) Is $f(x)$ continuous at $x = 1$? Explain.

8. If

$$f(x) = \begin{cases} x^3 + 6 & \text{if} & x < -1 \\ x^3 + 4 & \text{if} & x \geq -1 \end{cases}$$

Is $f(x)$ continuous at $x = -1$? Explain.

9. Find values of $a$ and $c$ if $f(x)$ is continuous at $x = 4$.

$$f(x) = \begin{cases} 2cx & \text{if} & x < 4 \\ x + a & \text{if} & x = 4 \\ x^2 - 6 & \text{if} & x > 4 \end{cases}$$

10. Find the values of constants $c$ and $k$ that make $f(x)$ continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} x + 2c & \text{if} & x < -2 \\ 3cx + k & \text{if} & -2 \leq x \leq 1 \\ 3x - 2k & \text{if} & x > 1 \end{cases}$$

11. Let

$$f(x) = \begin{cases} 3x^2 - 2x + 4 & \text{if} & x < -1 \\ -9x & \text{if} & -1 \leq x < 2 \\ 3x - 4 & \text{if} & x \geq 2 \end{cases}$$

   (a) Prove $f(x)$ is continuous at $x = -1$.

   (b) $f(x)$ is discontinuous at $x = 2$. State which continuity property fails.

12. Find all real numbers $c$ for which the function $g(x)$ given by

$$g(x) = \begin{cases} x^2 & \text{if} & x < \frac{1}{2} \\ c - x^2 & \text{if} & x \geq \frac{1}{2} \end{cases}$$

is continuous at $x = \frac{1}{2}$.

13. Let

$$f(x) = \begin{cases} x^2 & \text{if} & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1 & \text{if} & x = 0 \\ 2x - 1 & \text{if} & x < -1 \text{ or } x > 1 \end{cases}$$

   (a) Discuss the continuity of $f(x)$ at $x = -1, 1, 0$.

   (b) Sketch the graph of $f(x)$.

14. Prove, using the definition of continuity, that

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 7 & \text{if} \quad x = 2 \end{cases}$$

is discontinuous at $x = 2$.

15. Find a value for the constant $k$ that will make the functions continuous at the indicated value of $x$.

    (a) At $x = 1$

    $$f(x) = \begin{cases} 7x - 2 & \text{if} \quad x \leq 1 \\ kx^2 & \text{if} \quad x > 1 \end{cases}$$

    (b) At $x = 0$

    $$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \neq 0 \\ k & \text{if} \quad x = 0 \end{cases}$$

16. If

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \leq 5 \\ 7 - x & \text{if} \quad x > 5 \end{cases}$$

prove, using the definition of continuity, that the function $f(x)$ is discontinuous at $x = 5$.

17. If

$$g(x) = \begin{cases} 3x + 7 & \text{if} \quad x \leq 4 \\ kx - 1 & \text{if} \quad x > 4 \end{cases}$$

find the value of the constant $k$ that makes the function $g(x)$ continuous on $(-\infty, \infty)$.

18. If

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ x + 2 & \text{if} \quad 0 \leq x < 3 \\ 2x - 1 & \text{if} \quad x > 3 \end{cases}$$

    (a) Sketch the graph of $f(x)$.

    (b) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

    (c) Find $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$.

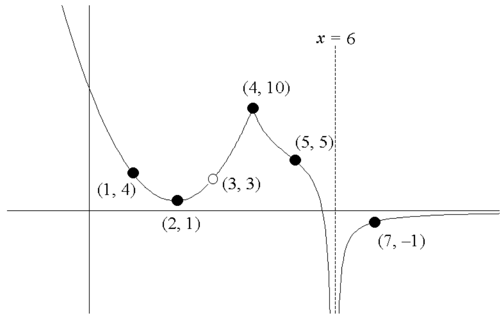
    (d) Is $f(x)$ continuous at $x = 0$? Explain.

    (e) Is $f(x)$ continuous at $x = 3$? Explain.

19. Find all values of $x$ at which the following function is discontinuous. Support your answer with specific references to the definition of continuity.

$$f(x) = \begin{cases} \dfrac{4x}{x+1} & \text{if} \quad x \leq 1 \\ 4 - 2x & \text{if} \quad 1 < x \leq 2 \\ 3x - 4 & \text{if} \quad x > 2 \end{cases}$$

20. Discuss (a) the continuity, and (b) the differentiability, of the function whose graph appears below




21. (a) Is the function

    $$f(x) = \begin{cases} x - 1 & \text{if} \quad x \neq 2 \\ 3 & \text{if} \quad x = 2 \end{cases}$$

    continuous at $x = 2$? Support your answer with a brief explanation.

    (b) Given $f(x) = |x|$

    i. Is $f$ continuous at $x = 0$? Why?

    ii. Is $f$ differentiable at $x = 0$? Why?

22. Sketch the graph of

$$f(x) = \begin{cases} 1 + x & \text{if} \quad x \leq -1 \\ 1 + x^2 & \text{if} \quad -1 < x < 2 \\ 7 - x & \text{if} \quad x \geq 2 \end{cases}$$

For what values of $x$ is $f(x)$ not continuous? Give reasons for your answer.

23. State whether the following are TRUE or FALSE and justify your answer briefly:

    (a) The function $f(x) = \dfrac{x^3 - x}{x}$ is continuous at $x = 0$.

    (b) The function $f(x) = \dfrac{x^3 - x}{x}$ is continuous at $x = 1$.

    (c) The function

    $$f(x) = \begin{cases} x^3 + 2x & \text{if} \quad x < -1 \\ |2x| - 5 & \text{if} \quad x \geq -1 \end{cases}$$

    is continuous at $x = -1$.

24. Let

$$f(x) = \begin{cases} x^2 + 2 & \text{if} \quad x < 1 \\ 4 - x & \text{if} \quad x \geq 1 \end{cases}$$

(a) Is this function continuous at $x = 1$?

(b) Use the definition of continuity to justify your answer.

25. If
$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x < 3 \\ x+4 & \text{if} \quad x \geq 3 \end{cases}$$

(a) Is $f(x)$ continuous at $x = 3$? Reason?

(b) Is $f(x)$ differentiable at $x = 3$? Reason?

26. If
$$g(x) = \begin{cases} 4x-1 & \text{if} \quad x \leq 2 \\ x^2+1 & \text{if} \quad x > 2 \end{cases}$$

(a) Is $g(x)$ continuous at $x = 2$? Reason?

(b) Is $g(x)$ differentiable at $x = 2$? Reason?

27. Discuss the continuity of the function
$$h(x) = \begin{cases} \dfrac{-1}{x+2} & \text{if} \quad x \leq -1 \\ x^3-1 & \text{if} \quad -1 < x \leq 2 \\ 9-x & \text{if} \quad x > 2 \end{cases}$$

28. Let
$$G(x) = \begin{cases} x^2+2x & \text{if} \quad x < -1 \\ 0 & \text{if} \quad -1 \leq x \leq 1 \\ \ln x & \text{if} \quad x > 1 \end{cases}$$

(a) Where is $G(x)$ not continuous?

(b) Where is $G(x)$ not differentiable?

## Answers:

1. The one-sided limits are different as we see below
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3$$
Therefore $\lim_{x\to 2} f(x)$ does not exist. Hence $f$ is not contiunous at $x = 2$.

2. 
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 = 3$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2$$
Therefore $\lim_{x\to 1} f(x)$ does not exist. Hence $f$ is not contiunous at $x = 1$.

3. (a) $f(1)$ is not defined.

(b) $\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} 2x = 2 \neq 1 = \lim_{x\to 1^+} x^2$

(c) $\lim_{x\to 1} f(x) \neq f(1)$ since $2 \neq \sqrt{1}$.

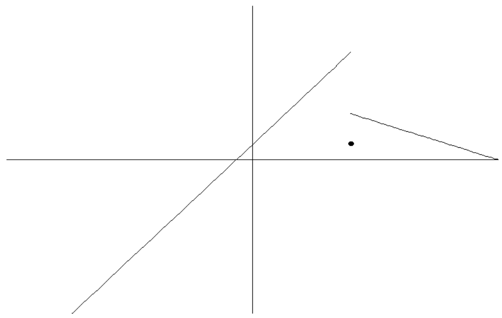
4. (a) $f(-2) = -5$

(b) $\lim_{x\to -2} \dfrac{x^2-x-6}{x+2}$
$= \lim_{x\to -2} \dfrac{(x-3)(x+2)}{(x+2)}$
$= \lim_{x\to -2} (x-3) = -5$

(c) $\lim_{x\to -2} f(x) = f(-2)$

5. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ if $\lim_{x\to 3^-} bx = \lim_{x\to 3^+} 6$, i.e. if $b \cdot 3 = 6$. Therefore $b = 2$.

6. (a)



(b) i. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x+1) = 7$ and

ii. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (5-x) = 3$

therefore $\lim_{x\to 2} f(x)$ does not exist, and so $f$ is discontinuous at $x = 2$.

7. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x) = 2$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3+x) = 4$ therefore $\lim_{x\to 1} f(x)$ does not exist, and so $f$ is not continuous at $x = 1$.

8. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x^3+4) = 3$ and $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^3+6) = 5$ therefore $\lim_{x\to -1} f(x)$ does not exist, and so $f$ is discontinuous at $x = -1$.

9. If $f$ is continuous at $x = 4$ then
$$\lim_{x\to 4^-} f(x) = f(4) = \lim_{x\to 4^+} f(x)$$
Therefore
$$\lim_{x\to 4^-} 2cx = 4 + a = \lim_{x\to 4^+} (x^2-6)$$

which yields

$$8c = 4 + a = 10$$

Thus $c = \frac{5}{4}$ and $a = 6$.

10. At $x = -2$

$$\begin{aligned} \lim_{x\to-2^-} f(x) &= f(-2) &= \lim_{x\to-2^+} f(x) \\ \lim_{x\to-2^-} (x+2c) &= -6c+k &= \lim_{x\to-2^+} (3cx + \\ 8c - k &= -6c+k &= -6c+k \end{aligned}$$

Thus we obtain the equation

$$(1) \qquad 8c - k = 2$$

At $x = 1$

$$\begin{aligned} \lim_{x\to1^-} f(x) &= f(1) &= \lim_{x\to1^+} f(x) \\ \lim_{x\to1^-} (3cx+k) &= 3c+k &= \lim_{x\to1^+} (3x-2k) \\ 3c+k &= 3c+k &= 3-2k \end{aligned}$$

Here we obtain

$$(2) \qquad 3c + 3k = 3$$

Now we need only solve the system of equation (1) and (2) to get the solution:
$c = 1/3, k = 2/3$.

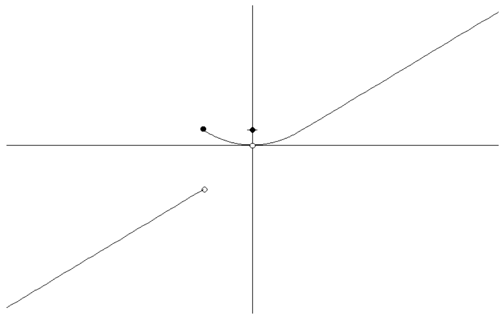
11. (a) $\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} (3x^2 - 2x + 4) = 9$
$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} (-9x) = 9$
Thus $\lim_{x\to-1} f(x) = 9 = f(-1)$
Hence $f$ is continuous at $x = -1$.

(b) $\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (-9x) = -18$
$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (3x-4) = 2$
Thus $\lim_{x\to2} f(x)$ does not exist
and so $f$ is not continuous at $x = 2$.

12. $\lim_{x\to 1/2^-} f(x) = f(1/2) = \lim_{x\to 1/2^+} f(x)$
$\lim_{x\to 1/2^-} x^2 = c - 1/4 = \lim_{x\to 1/2^+} c - x^2$
Which becomes $1/4 = c - 1/4 = c - 1/4$
Thus $c = 1/2$.

13. (a) i. $f$ is discontinuous at $x = -1$ since $\lim_{x\to-1} f(x)$ does not exist.
ii. $f$ is discontinuous at $x = 0$ since $f(0) \neq \lim_{x\to0} f(x)$.
iii. $f$ is continuous at $x = 1$.

(b)



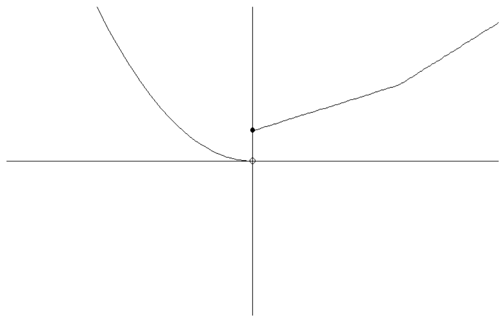
14. $\lim_{x\to2} f(x) = 4 \neq 7 = f(2)$,
therefore $f$ is discontinuous at $x = 2$.

15. (a) $k = 5$.
(b) $k = 1$.

16. $\lim_{x\to5^-} f(x) = \lim_{x\to5^-} (x+1) = 6$
$\lim_{x\to5^+} f(x) = \lim_{x\to5^+} (7-x) = 2$
Thus $\lim_{x\to5} f(x)$ does not exist
and so $f$ is not continuous at $x = 5$.

17. $k = 5$.

18. (a)



(b) $\lim_{x\to0^+} f(x) = 2$; $\lim_{x\to0^-} f(x) = 0$
(c) $\lim_{x\to3^+} f(x) = 5$; $\lim_{x\to3^+} f(x) = 5$
(d) No, since $\lim_{x\to0} f(x)$ does not exist.
(e) No, since $f(3)$ is not defined.

19. $f$ is not continuous at $x = -1$
since $f$ has a vertical asymptote there.
$f$ is not continuous at $x = 2$
since $\lim_{x\to2} f(x)$ does not exist.

20. (a) Continuous everywhere except at $x = 3$ and $x = 6$.

(b) Differentiable everywhere except at $x = 3$, $x = 4$, and $x = 6$.

21. (a) No, since $\lim_{x\to 2} f(x) = 1 \neq 3 = f(2)$.
    (b) i. Yes, since $\lim_{x\to 0} f(x) = 0 = f(0)$.
        ii. No, since $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ does not exist. [To see this show that the left- and right-limits are different ($-1$ and 1).]

22. $f$ is discontinuous at $x = -1$, since $\lim_{x\to -1} f(x)$ does not exist.

23. (a) No, since $f(0)$ is not deined.
    (b) Yes, since $\lim_{x\to 1} f(x) = 0 = f(1)$.
    (c) Yes, since $\lim_{x\to -1} f(x) = -3 = f(-1)$.

24. (a) Yes.
    (b) i. $f(1)$ is defined and equals 3.
        ii. $\lim_{x\to 1} f(x)$ exists and equals 3.
        iii. Thus $\lim_{x\to 1} f(x) = f(1)$.

25. (a) Yes, since $\lim_{x\to 3} f(x) = 7 = f(3)$.
    (b) No, since $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ does not exist. [To see this show that the left- and right-limits are different (2 and 1).]

26. (a) No, since $\lim_{x\to 2} g(x)$ does not exist.
    (b) No, since $g$ is not continuous at $x = 2$.

27. $h$ is not continuous at $x = -2$, since it has a vertical asymptote there.
    $h$ is not continuous at $x = -1$, since $\lim_{x\to -1} h(x)$ does not exist.
    $h$ is continuous everywhere else.

28. (a) $G$ is not continuous at $x = -1$.
    (b) $G$ is not differentiable at $x = -1$ and $x = 1$.