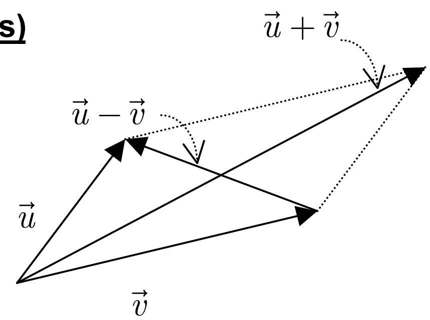
VECTORS (Formulas)



$\| \vec{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Unit vector along $\vec{u} = \dfrac{1}{\| \vec{u} \|}\vec{u}$

Vector of length k along $\vec{u} = \dfrac{k}{\| \vec{u} \|}\vec{u}$

Dot Product Properties

(1) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$ (2) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$ (3) $\vec{u} \bullet \vec{u} = \| \vec{u} \|^2$

(4) $\| \vec{u} + \vec{v} \|^2 = (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v}) = \vec{u} \bullet \vec{u} + 2\vec{u} \bullet \vec{v} + \vec{v} \bullet \vec{v} = \| \vec{u} \|^2 + 2\vec{u} \bullet \vec{v} + \| \vec{v} \|^2$

(5) $k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v}) = (k\vec{u}) \bullet \vec{v}$

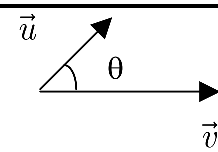
Cross Product Properties

(1) $\vec{u} \times \vec{O} = \vec{O}$ (2) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (3) $\vec{u} \times (k\vec{v}) = k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v}$

(4) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (5) $\vec{u} \times \vec{u} = \vec{O}$

(6) $\vec{u} \bullet (\vec{u} \times \vec{v}) = 0$ (scalar) and $\vec{v} \bullet (\vec{u} \times \vec{v}) = 0$ (scalar)

To find angle between 2 vectors



Use $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}$ or $\sin\theta = \dfrac{\vec{u} \times \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}$ (but $\cos\theta = \dfrac{\vec{u}.\vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}$ is better !)

To find angle in a Δ ABC: form vectors to find angle A then use cos A = $\dfrac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\|\overrightarrow{AB}\| \; \|\overrightarrow{AC}\|}$

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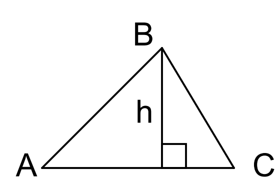
cosine (angle between 2 intersecting lines) = $\dfrac{\vec{d_1} \bullet \vec{d_2}}{\|\vec{d_1}\| \; \|\vec{d_2}\|}$

cosine (angle between a line and a plane) = $\dfrac{\vec{d} \bullet \vec{n}}{\|\vec{d}\| \; \|\vec{n}\|}$

cosine (angle between 2 planes) = $\dfrac{\vec{n_1} \bullet \vec{n_2}}{\|\vec{n_1}\| \; \|\vec{n_2}\|}$



To find the area of a parallelogram formed by vectors $\vec{u}$ and $\vec{v}$, find $\|\vec{u} \times \vec{v}\|$

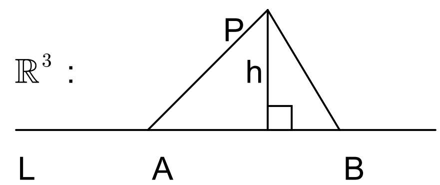
area of triangle formed by $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}\|\vec{u} \times \vec{v}\|$

to find area of triangle ABC : $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$

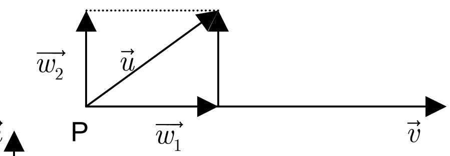
to find the perpendicular height from B to AC (for example): $\frac{1}{2}$(base) h = $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| \rightarrow$ h $= \dfrac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AC}\|}$

VECTORS (Formulas)

To find the distance from a point P to a line L containing points A and B in $\mathbb{R}^3$:

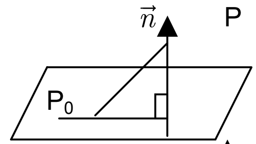


$\frac{1}{2}$(base) h = $\frac{1}{2}\left\|\overrightarrow{AP}\times\overrightarrow{AB}\right\| \rightarrow h = \frac{\left\|\overrightarrow{AP}\times\overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}$

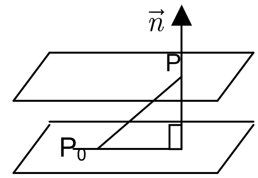


projection$_{\vec{v}}\ \vec{u} = \frac{\vec{u}\bullet\vec{v}}{\|\vec{v}\|^2}\vec{v} = \vec{w_1}$; $\vec{w_1} + \vec{w_2} = \vec{u} \Rightarrow \vec{w_2} = \vec{u} - \vec{w_1}$

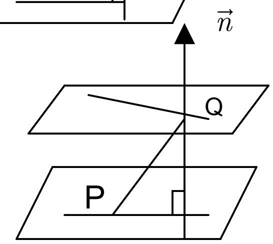
$\vec{w_1} =$ component of $\vec{u}$ along $\vec{v}$, $\vec{w_2} =$ component of $\vec{u}$ $\perp$ to $\vec{v}$



To find distance from a point to a plane = $\left\| \text{Proj}_{\vec{n}}\ \overrightarrow{P_0P} \right\|$



Distance between 2 parallel planes = $\left\| \text{Proj}_{\vec{n}}\ \overrightarrow{P_0P} \right\|$



To find distance between skew lines = $\left\| \text{Proj}_{\vec{n}}\ \overrightarrow{PQ} \right\|$

where $\vec{n} = \vec{d_1} \times \vec{d_2}$

To find the equation of a line in $\mathbb{R}^2$ or $\mathbb{R}^3$

vector form: where $a = \Delta x$, $b = \Delta y$, $c = \Delta z$ or parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ = a specific point + $t\,\vec{d}$

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$$

To find the equation of a plane in $\mathbb{R}^3$

$\vec{n} \bullet \left(x - x_1, y - y_1, z - z_1\right) = 0$ where $\vec{n} = \left(a, b, c\right)$, normal to the plane

$a\left(x - x_1\right) + b\left(y - y_1\right) + c\left(z - z_1\right) = 0$ (point-normal form)

$ax + by + cz + d = 0$ (standard form)

Scalar Triple Product $\vec{u} \bullet \left(\vec{v} \times \vec{w}\right) = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$

VECTORS (Formulas)

When are vectors :

(a) parallel ? when they are scalar multiples
(b) perpendicular ? when their dot product = 0
(c) equal ? when all corresponding components are identical

When are lines :

(a) parallel ? when there is no intersection and the $\vec{d}$'s are multiples
(b) perpendicular ? when there is a single intersection point and $\vec{d_1} \bullet \vec{d_2} = 0$
(c) equal ? when there is an infinite intersection and $\vec{d}$'s are multiples

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Volume of parallelepiped determined by $\vec{u}, \vec{v}$ and $\vec{w} = \left| \vec{w} \bullet \vec{u} \times \vec{v} \right|$ or absolute value of

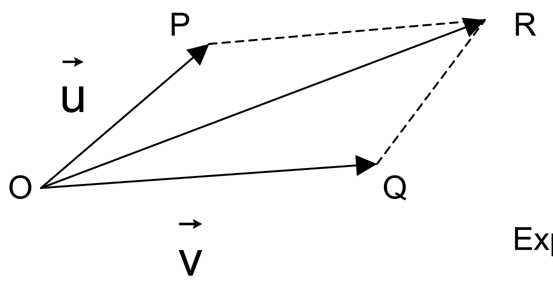
$\det \begin{pmatrix} \vec{w} \\ \vec{v} \\ \vec{u} \end{pmatrix}$ ← order of vectors is irrelevant since the <u>absolute</u> value is being used.

$\vec{u} \bullet \vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix}$ ← Here the order of the vectors is irrelevant – we are not taking the absolute value.

Relationship between $\vec{u} \times \vec{v}$ and the angle between $\vec{u}, \vec{v}$: $\left\| \vec{u} \times \vec{v} \right\| = \left\| \vec{u} \right\| \ \left\| \vec{v} \right\| \ \sin\theta$

Vectors Problems

(1) Let $\vec{u}$ and $\vec{v}$ be the position vectors of points P and Q respectively and let R be the terminal point of $\vec{u} + \vec{v}$



$O =$ the origin, $\overrightarrow{OP} = \vec{u}$, $\overrightarrow{OQ} = \vec{v}$

Express the following in terms of $\vec{u}$ and $\vec{v}$:

(a) $\overrightarrow{QP}$; (b) $\overrightarrow{PQ}$; (c) $\overrightarrow{RP}$; (d) $\overrightarrow{QR}$; (e) $\overrightarrow{RQ}$; (f) $\overrightarrow{RO}$

(2) Determine whether $\vec{u}$ and $\vec{v}$ are parallel or not .

(a) $\vec{u} = (1, 2, -1)$, $\vec{v} = (2, 1, 0)$ (b) $\vec{u} = (3, -6, 3)$, $\vec{v} = (-1, 2, -1)$

(c) $\vec{u} = (1, 0, 1)$, $\vec{v} = (-1, 0, 1)$ (d) $\vec{u} = (2, 0, -1)$, $\vec{v} = (-8, 0, 4)$

(3) Find a point Q such that $\overrightarrow{PQ}$ has

(i) the same direction and the same magnitude as $\vec{v}$ (i.e. $\overrightarrow{PQ} = \vec{v}$)

(ii) the opposite direction and the same magnitude as $\vec{v}$ (i.e. $\overrightarrow{PQ} = -\vec{v}$)

(a) P (-1 , 2 , 2) , $\vec{v} = (1, 2, -1)$ (b) P (3 , 0 , -1) , $\vec{v} = (2, -1, 3)$

(4) If $\vec{u} = (3, -1, 0)$, $\vec{v} = (4, 0, 1)$, $\vec{w} = (1, 1, 3)$, find $\vec{x}$ such that :

(a) $3(2\vec{u} + \vec{x}) + \vec{w} = 2\vec{x} - \vec{v}$ (b) $2(3\vec{v} - \vec{x}) = 5\vec{w} + \vec{u} - 3\vec{x}$

(5) Find c_1, c_2, c_3 (scalars) such that

(a) $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = (2, -1, 6)$ (b) $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = (1, 3, 0)$

where $\vec{u} = (1, 1, 2)$, $\vec{v} = (0, 1, 2)$, $\vec{w} = (1, 0, -1)$

(6) Let $\vec{u} = (3, -1, 0)$, $\vec{v} = (4, 0, 1)$, $\vec{w} = (1, 1, 1)$, show that there does not exist c_1, c_2, c_3 (scalars) such that :

(a) $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = (1, 2, 1)$ (b) $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = (5, 6, -1)$

(7) Let $P_1 = (2, 1, -2)$, $P_2 = (1, -2, 0)$. Find the coordinates of P such that

(a) P is 1/5 of the way from P_1 to P_2 (b) P is 1/4 of the way from P_1 to P_2

(c) P is 1/2 of the way from P_1 to P_2 (i.e. the midpoint)

Note : P is a point on the vector $\overrightarrow{P_1 P_2}$

Answers : (1 a) $\vec{u} - \vec{v}$; (1 b) $\vec{v} - \vec{u}$; (1 c) $-\vec{v}$; (1 d) $\vec{u}$; (1 e) $-\vec{u}$; (1 f) $-\vec{u} - \vec{v}$; (2 a) no ; (2 b) yes, $\vec{u} = -3\vec{v}$; (2 c) no
(2 d) yes, $\vec{v} = -4\vec{u}$; (3 a) (i) (0 , 4 , 1) , (ii) (-2 , 0 , 3) ; (3 b) (i) (5 , -1 , 2) , (ii) (1 , 1 , -4)
(4 b) (-16 , 4 , 9) ; (5 b) $c_1 = -5$, $c_2 = 8$, $c_3 = 6$; (7 a) P (9/5 , 2/5 , -8/5) ;
(7 b) P (7/4 , 1/4 , -3/2) ; (7 c) P (3/2 , -1/2 , -1)

Text : Ex 3.1 (1 – 11) and Ex 3.2 (1 – 6)