THEORY

1.

$$A \sim B \Rightarrow B \sim A$$

$A \sim B \Rightarrow \exists \underbrace{E_1 E_2 ... E_n}_{\text{elementary matrices}}$ such that $E_n ... E_2 E_1 A = B \Rightarrow A = \left(E_n ... E_2 E_1\right)^{-1} B = E_1^{-1} E_2^{-1} ... E_n^{-1} A$

when E_i^{-1} is an elementary matrix (for all i) (inverse of an elementary is elementary) $\Rightarrow B \sim A$

2.

$$A \sim B \, , B \sim C \Rightarrow A \sim C$$

$A \sim B \Rightarrow \exists \underbrace{E_1 E_2 ... E_n}_{\text{elementary matrices}}$ such that $E_n ... E_2 E_1 A = B \Rightarrow A = \left(E_n ... E_2 E_1\right)^{-1} B = E_1^{-1} E_2^{-1} ... E_n^{-1} A$

$B \sim C \Rightarrow \exists \underbrace{F_1 F_2 ... F_k}_{\text{elementary matrices}}$ such that $F_k ... F_2 F_1 B = C$

such that $\underbrace{F_k ... F_2 F_1 \underbrace{E_n ... E_2 E_1 A}_{\text{substituting for } B}}_{\text{all elementary matrices except } A} = C \Rightarrow A \sim C$

Equivalences   Let A be an nxn square matrix and R=RREF of A.

$(1)\ A^{-1} \exists$ $(2)\ A\vec{x} = \vec{b}$ has a unique solution

$(3)\ A\vec{x} = \vec{0}$ has a unique solution (4) Rank $A =$ # leading ones in R = n

$(5)\ A \sim \mathrm{I}$ $(6)\ A =$ a product of elementary matrices (7) det $A \neq 0$

"Alternate" list of Equivalent Statements

$(1)\ A^{-1} \nexists$ $(2)\ A\vec{x} = \vec{b}$ has a parametric solution or is inconsistent

$(3)\ A\vec{x} = \vec{0}$ has a parametric solution (hence, nontrivial solutions) (4) Rank $A <$ n

$(5)\ A \sim R$ (where R has at least one row of zeros)

$(6)\ A$ cannot be written as a product of elementary matrices (7) det $A = 0$

Problems - Anton: Ex 2.1 9, 12, 13, 18, 19
Ex 2.2 2, 3, 4, 7, 10, 12, 13, 17
Ex 2.3 2, 3, 4, 5, 6-12, 13
Ex 2.4 1, 2, 4, 5, 7, 9, 10, 11, 12, 22, $\underbrace{\text{16-18, 23}}_{\text{Cramer's Rule!}}$, $\underbrace{\text{25, 26}}_{\text{theory}}$
Ex 1.6 ; 24, 25 (theory) , Ex 1.5 ; 19, 20 (theory)
Note in (20) A is singular means A is square and $A^{-1} \nexists$

(1) If $\vec{x_1}$ is a solution to $A\vec{x} = \vec{b}$ and $\vec{x_2}$ is a solution to $A\vec{x} = \vec{0}$, show that $\vec{x_1} + k\vec{x_2}$ is a solution to $A\vec{x} = \vec{b}$.

(2) Let A, B, P represent $n \times n$ invertible matrices (i.e. $A^{-1}, B^{-1}, P^{-1} \exists$) and $B = PAP^{-1}$

(a) show that $B^2 = PA^2P^{-1}$ (b) show that $B^{-1} = PA^{-1}P^{-1}$

$(3)\ (a)$ If $A^2 = \mathrm{I}$, prove det $A = \pm 1$
(b) If $A^2 = A$, prove det $A = 0$ or 1
(Take the det of both sides)

$(4)\ A_{3\times3}$; $B_{3\times3}$ such that det $\left(3A^{-1}\right) = -5$ and det $\left(A^2 B^{-1}\right) = -5$. Find det A ; det B.

(Answer: det $A = -\frac{27}{5}$; det $B = -\frac{(27)^2}{125}$)