Review Problems # 5

(1) Determine whether the following sets are LI or LD. Justify your answers.

(a) $S_1 = \{(2,1,-4)\}$; (b) $S_2 = \{(2,1,-4),(0,0,0)\}$

(c) $S_3 = \{(3,1,2),(6,2,4)\}$; (d) $S_4 = \{(1,0,1,1),(2,1,0,3),(1,-1,2,5)\}$

(2) Write (if possible) $\vec{u}$ as a L.C. of $\vec{v}$ and $\vec{w}$ where
$\vec{u} = (11,3,22,11)$, $\vec{v} = (-1,2,3,4)$ and $\vec{w} = (3,-1,2,-1)$

(3) Determine if $\vec{v} = (1,-3,2)$ can be written as a L.C. of

$\vec{u_1} = (1,2,-1)$, $\vec{u_2} = (3,5,2)$ and $\vec{u_3} = (4,7,1)$

(4) Find a basis for each of the subspaces
(a) $\{(x,y,z) \in \mathbb{R}^3 | 2x + y - 3z = 0\}$; (b) $\{(x,y,z,w) \in \mathbb{R}^4 \,|\, x + 5y = 0, z = 0, w = 0\}$

(5) Does $(8,-3,-6)$ belong to span $\{(1,0,3),(-2,1,4)\}$? Why or why not ?

(6) Determine the span of each of the following sets of vectors

(a) $\left\{\begin{pmatrix}1\\2\\-5\end{pmatrix},\begin{pmatrix}3\\1\\1\end{pmatrix},\begin{pmatrix}4\\3\\-4\end{pmatrix}\right\}$; (b) $\left\{\begin{pmatrix}1\\-1\\3\end{pmatrix},\begin{pmatrix}2\\-2\\6\end{pmatrix},\begin{pmatrix}3\\-3\\9\end{pmatrix}\right\}$; (c) $\left\{\begin{pmatrix}1\\2\\2\\0\end{pmatrix},\begin{pmatrix}2\\-1\\0\\3\end{pmatrix},\begin{pmatrix}0\\0\\1\\2\end{pmatrix}\right\}$

(7)

$$\text{Let } A = \begin{pmatrix}1 & 2 & 2 & 0 & 1\\ 2 & 4 & 0 & 3 & 1\\ 5 & 10 & 6 & 3 & 4\\ 2 & 4 & 4 & 0 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 0 & \frac{3}{2} & \frac{1}{2}\\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} = R$$

find : (a) the Null Space of A , a basis for it and the dimension
(b) a basis for Column Space of A and the dimension
(c) write the non-basic column vectors as L.C.'s of Basic Column vectors
(d) a basis for Row Space of A and the dimension

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(8) Let $\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}$ represent the column vectors of

$$A = \begin{pmatrix} 1 & -2 & 4 & -5 \\ 2 & 3 & 7 & 4 \\ 0 & 3 & 4 & 6 \\ 1 & 4 & 6 & 7 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$, the REEF of A

(a) Express $\vec{u_4}$ as a L.C. of $\vec{u_1}, \vec{u_2}, \vec{u_3}$

(b) which one of the 4 vectors is not a L.C. of the other three?

(c) Are the column vectors of A LI or LD ? Justify

(d) Do the column vectors of A span $\mathbb{R}^4$? Why or why not ?

(e) Find a basis for the Column Space of A

(f) Find a basis for the Null Space of A

(g) Find a basis for Row Space of A

(9) Find the solution space of $\begin{cases} x_1 - x_2 + x_3 - 2x_4 = 0 \\ x_1 + x_2 - 2x_3 + x_4 = 0 \end{cases}$

(10) Show that span $\{(1,0,-3),(2,-1,4)\}$ = span $\{(5,-2,5),(3,-1,1)\}$

(11) (a) Write $\begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}$ as a L.C. of $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

(b) Does $S_1 = \left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \right\}$ span $\mathbb{R}^3$? Explain

(c) Find a set S_2 of 3 vectors in $\mathbb{R}^3$ which does span $\mathbb{R}^3$. (Justify your choice)

(12) Find what values of c is the set of vectors $\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix} \right\}$ LD ?

(13) Let $\vec{v_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}; \vec{v_2} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}; \vec{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \vec{v_4} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(a) Show that $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is LD

(b) Explain why these vectors do not span $\mathbb{R}^4$

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(14) Prove that the following subsets are not subspaces of the given vector space

(a) $S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| 5x = 2yz \right\}$; (b) $S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \middle| x = t+1, y = t-3, z = t, w = t \right\}$

(c) $S_3 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \middle| xyzw = 0 \right\}$; (d) $S_4 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \middle| 3xy = 4zw \right\}$; (e) $S_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x \geq 0, y \geq 0, z \geq 0 \right\}$

(15) Prove that the following are subspaces of the given vector space

(a) $S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x + 3y - z = 0 \right\}$; (b) $S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x - 2z = 0 \right\}$; (c) $S_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x = 3y \right\}$

(16) Find a basis for subspaces in # 15 . State the dimension of each.

(17) (a) Prove $S = \text{span}\left\{ \vec{v_1}, \vec{v_2}, \vec{v_3} \right\}$ is a subspace of $\mathbb{R}^4$ where $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are vectors in $\mathbb{R}^4$

(b) Is the dimension of S = 3 ? (Justify)

Answers

(1 a) LI ; $c_1(2,1,-4) = (0,0,0) \Rightarrow c_1 = 0$; (1 b) LD ; $0(2,1,-4) + \underbrace{5}_{\text{non-zero}}(0,0,0) = (0,0,0)$

(1 c) LD ; $2(3,1,2) - (6,2,4) = (0,0,0)$; (1 d) LI ; $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \overset{c_1}{1} & \overset{c_2}{0} & \overset{c_3}{0} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $c_1 = c_2 = c_3 = 0$ uniquely

(2) $\vec{u} = 4\vec{v} + 5\vec{w}$

(3) $\vec{v}$ is not a L.C. of $\vec{u}$'s $\rightarrow \left[\begin{array}{ccc|c} \overset{c_1}{1} & \overset{c_2}{3} & \overset{c_3}{4} & 1 \\ 2 & 5 & 7 & -3 \\ -1 & 2 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & \neq 0 \end{array}\right]$

(4 a) $\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$ is one possibility ; (4 b) $\left\{ (-5,1,0,0) \right\}$ is one possibility

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Answers

(5) yes! $2\begin{pmatrix}1\\0\\3\end{pmatrix}-3\begin{pmatrix}-2\\1\\4\end{pmatrix}=\begin{pmatrix}8\\-3\\-6\end{pmatrix}\Rightarrow\left(8,-3,-6\right)$ is a L.C. of the 2 vectors

(6 a) plane in $\mathbb{R}^3$ with equation $7x-16y-5z=0$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}=t\begin{pmatrix}1\\2\\-5\end{pmatrix}+t\begin{pmatrix}3\\1\\1\end{pmatrix}$ still a plane in $\mathbb{R}^3$

(6 b) line in $\mathbb{R}^3$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=t\begin{pmatrix}1\\-1\\3\end{pmatrix}$; (6 c) LI vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=r\begin{pmatrix}1\\2\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\-1\\0\\3\end{pmatrix}+t\begin{pmatrix}0\\0\\1\\2\end{pmatrix}$ a 3-d subspace of $\mathbb{R}^4$

(7 a) $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-2s-\frac{3}{2}t-\frac{1}{2}r\\s\\\frac{3}{4}t-\frac{1}{4}r\\t\\r\end{pmatrix}=s\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}-\frac{3}{2}\\0\\\frac{3}{4}\\1\\0\end{pmatrix}+r\begin{pmatrix}-\frac{1}{2}\\0\\-\frac{1}{4}\\0\\1\end{pmatrix}$; (7 b) $\left\{\begin{bmatrix}1\\2\\5\\2\end{bmatrix},\begin{bmatrix}2\\0\\6\\4\end{bmatrix}\right\}$; d = 2

basis : $\left\{\left(-2,1,0,0,0\right),\left(-\frac{3}{2},0,\frac{3}{4},1,0\right),\left(-\frac{1}{2},0,-\frac{1}{4},0,1\right)\right\}$; d=3

(7 c) $\begin{pmatrix}2\\4\\10\\4\end{pmatrix}=2\begin{pmatrix}1\\2\\5\\2\end{pmatrix};\begin{pmatrix}0\\3\\3\\0\end{pmatrix}=\frac{3}{2}\begin{pmatrix}1\\2\\5\\2\end{pmatrix}-\frac{3}{4}\begin{pmatrix}2\\0\\6\\4\end{pmatrix};\begin{pmatrix}1\\1\\4\\2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\2\\5\\2\end{pmatrix}+\frac{1}{4}\begin{pmatrix}2\\0\\6\\4\end{pmatrix}$; (7 d) $\left\{\left(1,2,2,0,1\right),\left(2,4,0,3,1\right)\right\}$ or $\left\{\left(1,2,0,\frac{3}{2},\frac{1}{2}\right),\left(0,0,1,-\frac{3}{4},\frac{1}{4}\right)\right\}$

(8 a) $\vec{u_4}=-\vec{u_1}+2\vec{u_2}+0\vec{u_3}$; (8 b) $\vec{u_3}$ (it cannot be isolated since it has a “0” coefficient)

(8 c) LD (solution for c's is parametric ; $c_4=t$)

(8 d) No! Span { col vectors of A } is a 3-d subspace of $\mathbb{R}^4$, not the entire space

(8 e) $\left\{\vec{u_1},\vec{u_2},\vec{u_3}\right\}$;

(8 f) $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}t\\-2t\\0\\t\end{pmatrix}=t\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}$; basis = $\left\{\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}\right\}$; (8 g) $\left\{\left(1,-2,4,-5\right),\left(2,3,7,4\right),\left(0,3,4,6\right)\right\}$ or first 3 row vectors of R

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Answers

(9) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$; a 2-d subspace of $\mathbb{R}^4$ (same as Null Space A)

(10) both are planes whose equation is $3x + 10y + z = 0$

(11 a) $3\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}$;

(11 b) no! span $\left\{\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}\right\} =$ span $\left\{\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right\}$,

$\therefore$ a plane in $\mathbb{R}^3$, not the entire space

(11 c) any 3 LI vectors will span $\mathbb{R}^3$; for example $\{(1,0,0),(0,1,0),(0,0,1)\}$; check det $A \neq 0$

(12) det $\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & c \end{pmatrix} = (-1)(c-2) - 1(1) = -c + 1 = 0$ when $c = 1$; if det $A = 0$, $A \sim R$ where R has a row of zeros $\Rightarrow$ parametric solution for c's $\Rightarrow$ vectors are LD

(13 a) det $\left(\vec{v_1}\ \vec{v_2}\ \vec{v_3}\ \vec{v_4}\right) = 0 \Rightarrow$ vectors are LD i.e. related by a dependency equation

(13 b) you need 4 LI vectors to span the entire $\mathbb{R}^4$

(14 a) addition fails: $\vec{u} = \underbrace{\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}}_{5(2)=2(5)(1)} \in S_1$; $\vec{v} = \underbrace{\begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}}_{5(6)=2(3)(5)} \in S_1$ but $\vec{u} + \vec{v} = \underbrace{\begin{pmatrix} 8 \\ 8 \\ 6 \end{pmatrix}}_{5(8)\neq 2(8)(6)} \notin S_1$

(14 b) $(0,0,0,0) \notin S_2$ since $x = t + 1 = 0 \Rightarrow t = -1$, $y = t - 3 = 0 \Rightarrow t = 3$, $z = t = 0$, $w = t = 0$ contradictory t's ; also you could show addition or scalar multiplication fails

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Answers

(14 c) addition fails: $\underset{(1)(2)(0)(3)=0}{\vec{u}=\begin{pmatrix}1\\2\\0\\3\end{pmatrix}}\in S_3$; $\underset{(0)(1)(2)(3)=0}{\vec{v}=\begin{pmatrix}0\\1\\2\\3\end{pmatrix}}\in S_3$ but $\underset{(1)(3)(2)(6)\neq 0}{\vec{u}+\vec{v}=\begin{pmatrix}1\\3\\2\\6\end{pmatrix}}\notin S_3$

(14 d) addition fails: $\underset{3(1)(0)=(4)(2)(0)}{\vec{u}=\begin{pmatrix}1\\0\\2\\0\end{pmatrix}}\in S_4$; $\underset{3(4)(1)=(4)(1)(3)}{\vec{v}=\begin{pmatrix}4\\1\\1\\3\end{pmatrix}}\in S_4$ but $\underset{3(5)(1)\neq(4)(3)(3)}{\vec{u}+\vec{v}=\begin{pmatrix}5\\1\\3\\3\end{pmatrix}}\notin S_4$

(14 e) addition works: $\underset{x\geq 0,\,y\geq 0,\,z\geq 0}{\vec{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}}\in S_5$; $\underset{p\geq 0,\,q\geq 0,\,r\geq 0}{\vec{v}=\begin{pmatrix}p\\q\\r\end{pmatrix}}\in S_5$ and $\underset{x+p\geq 0,\,y+q\geq 0,\,z+r\geq 0}{\vec{u}+\vec{v}=\begin{pmatrix}x+p\\y+q\\z+r\end{pmatrix}}\in S_5$

scalar multiplication fails: $(1,2,3)\in S_5$ but $-2(1,2,3)=(-2,-4,-6)\notin S_5$

(15 a) Let: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3s+t\\s\\t\end{pmatrix}=s\begin{pmatrix}-3\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\1\end{pmatrix}=s\,\vec{a_1}+t\,\vec{a_2}$; where basis vectors are $\vec{a_1}=\begin{pmatrix}-3\\1\\0\end{pmatrix}$ and $\vec{a_2}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

Let: $\vec{u}=a\,\vec{a_1}+b\,\vec{a_2}\in S_1$ and $\vec{v}=c\,\vec{a_1}+d\,\vec{a_2}\in S_1$

$\left.\begin{cases}\text{addition : } \vec{u}+\vec{v}=(a+c)\vec{a_1}+(b+d)\vec{a_2}\in S_1\\ \text{scalar multiplication : } k\vec{u}=(ka)\vec{a_1}+(kb)\vec{a_2}\in S_1\end{cases}\right\}\rightarrow$ S_1 is a subspace of $\mathbb{R}^3$

(15 b) Let: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2t\\s\\t\end{pmatrix}=t\begin{pmatrix}2\\0\\1\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix}=t\,\vec{a_1}+s\,\vec{a_2}$; where basis vectors are $\vec{a_1}=\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\vec{a_2}=\begin{pmatrix}0\\1\\0\end{pmatrix}$

proof: same as in (15 a)

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Answers

(15 c) Let: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t \\ -t \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \end{pmatrix} = t\,\vec{a_1}$; where basis vector is $\vec{a_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Let: $\vec{u} = a\,\vec{a_1} \in S_3$ and $\vec{v} = b\,\vec{a_1} \in S_3$

$$\left\{ \begin{array}{l} \text{addition : } \vec{u} + \vec{v} = (a+b)\,\vec{a_1} \in S_3 \\ \text{scalar multiplication : } \text{k}\vec{u} = (\text{k}a)\,\vec{a_1} \in S_3 \end{array} \right\} \rightarrow S_3 \text{ is a subspace of } \mathbb{R}^3$$

(16 a) $\{(-3,1,0),(1,0,1)\}$, d = 2 ; (16 b) $\{(2,0,1),(0,1,0)\}$, d = 2 ; (16 c) $\{(3,1)\}$, d = 1

(17 a) Let: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = r\,\vec{v_1} + s\,\vec{v_2} + t\,\vec{v_3} \in S$

$$\left\{ \begin{array}{l} \vec{u} = a\,\vec{v_1} + b\,\vec{v_2} + c\,\vec{v_3} \in S \\ \vec{v} = c\,\vec{v_1} + d\,\vec{v_2} + e\,\vec{v_3} \in S \\ \vec{u} + \vec{v} = (a+c)\,\vec{v_1} + (b+d)\,\vec{v_2} + (c+e)\,\vec{v_3} \in S \\ \text{and } \text{k}\vec{u} = (\text{k}a)\,\vec{v_1} + (\text{k}b)\,\vec{v_2} + (\text{k}c)\,\vec{v_3} \in S \end{array} \right. \rightarrow S \text{ is a subspace of } \mathbb{R}^4$$

(17 b) No! dimension of *S* = 3 only if $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is LI and we don't know that.

Any of these options is possible:

The dimension is either 0 (if all vectors are $\vec{O}$,
1 (if all vectors are multiples) ,
2 (if $\vec{v_3} =$ a L.C. of $\vec{v_1}$ and $\vec{v_2}$) ,
or 3 (if vectors are LI)