

Review Problems # 3

(1) Given $E = \begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ d & e & f \end{pmatrix}$; $F = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \\ d & e & f \end{pmatrix}$; $K = \begin{pmatrix} 2a & 2b & 2c \\ 3 & 6 & 9 \\ d & e & f \end{pmatrix}$, if $\det E = 31$,

find (a) $\det F$; (b) $\det K$; (c) $\det (2E)$; (d) $\det (E^{-1})$; (e) $\det (EF)$; (f) $\det (K - E)$

(2) Let $A = \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$ and $\det A = 2$;

(i) find: (a) $\det (AA^t)$; (b) $\det (-A^2)$; (c) $\det (3A)^{-1}$; (d) $\det \begin{pmatrix} a & b & c \\ 3x & 3y & 3z \\ r + 2x & s + 2y & t + 2z \end{pmatrix}$; (e) C_{32}

(ii) what is the solution to $A\vec{x} = \vec{0}$?

(iii) What is the rank of A ?

(3) Let $A = \begin{pmatrix} 3 & 0 & -6 \\ -4 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}$

(a) find $\det A$, $\text{adj } A$ and A^{-1} , $(A)(\text{adj } A)$, $\det (\text{adj } A)$

(b) find A^{-1} using the $[A | I]$ process

(c) use A^{-1} to $AX = B$ where $B = \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}$

(d) find x, y, z using Cramer's Rule

(4) Evaluate $\det \begin{pmatrix} 3 & 1 & -2 & 4 \\ 1 & -2 & -3 & 2 \\ -2 & 4 & 8 & 3 \\ 0 & 2 & -2 & 4 \end{pmatrix}$; (5) Given $C = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 4 & 6 & 0 & -2 \\ 3 & -1 & 0 & 5 \\ -2 & 2 & -4 & 3 \end{pmatrix}$, find (a) C_{43} , (b) $\det C$

(6) Let $A = \begin{pmatrix} 2 & -2 & 4 \\ 5 & 1 & 3 \end{pmatrix}$; $B = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{7}{6} \end{pmatrix}$

(a) Find E_1, E_2, E_3 (elementary matrices such that $E_3 E_2 E_1 A = B$)

(b) Find F_1, F_2, F_3 such that $F_3 F_2 F_1 B = A$

(7) Find the following by inspection.

(a) $\det \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 5 & 7 & -3 & 0 \\ 2 & 9 & 8 & 5 \end{pmatrix}$; (b) $\det \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\det \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 7 & 8 \\ -1 & -2 & 2 & 3 \\ 4 & 8 & 9 & 8 \end{pmatrix}$

Review Problems # 3

(8) Let $A = \begin{pmatrix} 1 & 3 & 10k \\ 2 & 1 & -5 \\ 1 & -2 & 7 \end{pmatrix}$, (a) find K such that $\det A = 0$
 (b) find K such that $AX = 0$ has nontrivial solutions
 (c) find K such that $AX = B$ has a unique solution

(9) Let $A^{-1} = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$; $B^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$, find (a) A , (b) $(AB)^{-1}$, (c) $(AB)^t$, (d) A^2 ; (e) B^{-2}

(10) Given $A = \begin{pmatrix} 2 & 4 \\ -1 & -1 \end{pmatrix}$, find $A^{-2} - 3A + 4I$; (11) Find A if $(I + A)^{-1} = 3 \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$

(12) Simplify: (a) $(3C^{-1}AB)^{-1}(C^{-1}A^2)$, (b) $\det(3C^{-1}A^tB)$; (c) $(3C^{-1}A^tB)^t$

Assume A, B, C are 3×3 invertible matrices

(13) (a) $ABX = A + B$ for X assuming $A^{-1}, B^{-1} \exists$; (b) $AX + B = X$, assuming $(I - A)^{-1} \exists$

(14) Let $A = \begin{pmatrix} 0 & -5 \\ 2 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$, show (a) $(A + B)^{-1} \neq A^{-1} + B^{-1}$, (b) $A^2 - B^2 \neq (A - B)(A + B)$

(15) If $A^2 = A^t$, find all possible values for $\det A$; (16) Prove $(KA)^{-1} = \frac{1}{k} A^{-1}$

(17) Find A, B such that $A^{-1}, B^{-1} \exists$ but $(A + B)^{-1} \nexists$

(18) Find A, B such that $\det(A + B) \neq \det A + \det B$

Answers:

(1 a) -31 ; (b) 186 ; (c) 248 ; (d) $\frac{1}{31}$; (e) $(\det E)(\det F) = -961$; (f) 0

(2 i) (a) 4 ; (b) $(-1)^3(\det A)^2 = -4$; (c) $\frac{1}{54}$; (d) -6 ; (e) $-M_{32} = -(at - cr)$

(2 ii) $X = 0$, trivial solution only since A^{-1} exists;

(2 iii) rank $A = 3$ since A is reducible to I and I has 3 leading ones

(3 a) $\det A = 9$; $\text{adj } A = \begin{pmatrix} -3 & -6 & -6 \\ 9 & 12 & 21 \\ -3 & -3 & -3 \end{pmatrix}$; $A^{-1} = \frac{1}{9} \begin{pmatrix} -3 & -6 & -6 \\ 9 & 12 & 21 \\ -3 & -3 & -3 \end{pmatrix}$; $A(\text{adj } A) = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$;

$\det(\text{adj } A) = 9^2 = 81$; (3 b) $A^{-1} = \frac{1}{9} \begin{pmatrix} -3 & -6 & -6 \\ 9 & 12 & 21 \\ -3 & -3 & -3 \end{pmatrix}$ as in parts (a)

(3 c) $X = A^{-1}B = \begin{pmatrix} -8 \\ 30 \\ -5 \end{pmatrix}$; (d) same as (c); (4) -260 ; (5 a) 44 ; (5 b) -524

Review Problems # 3

Answers:

(6) $E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ is one possibility ; $A = (E_3 E_2 E_1)^{-1} B = E_1^{-1} E_2^{-1} E_3^{-1} B$

$E_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$; (7 a) -120 (Δ matrix) ;

(7 b) $(-1)(-2) = 2$ (row interchange \rightarrow diagonal matrix) ; (c) 0 (c_1 and c_2 are multiples)

(8 a) $k = -\frac{6}{5}$; (b) when $\det A = 0$, i.e. $k = -\frac{6}{5}$; (c) $\det A \neq 0$, $k \neq -\frac{6}{5}$

(9 a) $A = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$; (b) $\begin{pmatrix} 2 & -5 \\ 9 & -6 \end{pmatrix}$; (c) $\frac{1}{33} \begin{pmatrix} -6 & -9 \\ 5 & 2 \end{pmatrix}$; (d) $\frac{1}{121} \begin{pmatrix} -7 & 8 \\ -16 & 1 \end{pmatrix}$; (e) $\begin{pmatrix} 1 & -2 \\ 6 & -3 \end{pmatrix}$

(10) $\begin{pmatrix} -\frac{11}{4} & -13 \\ \frac{13}{4} & 7 \end{pmatrix}$; (11) $\begin{pmatrix} -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{4}{3} \end{pmatrix}$; (12 a) $\frac{1}{3} B^{-1} A^{-1} (C C^{-1}) A^2 = \frac{1}{3} B^{-1} A^{-1} (I A^2) = \frac{1}{3} B^{-1} A$;

(12 b) $\frac{27 (\det A) (\det B)}{\det C}$; (12 c) $3B^t A (C^{-1})^t$

(13 a) $X = B^{-1} + B^{-1} A^{-1} B$ or $B^{-1} (I + A^{-1} B)$ or $B^{-1} + (AB)^{-1} B$

(13 b) $X = (I - A)^{-1} B$ or $-(A - I)^{-1} B$; (14 a) $(A + B)^{-1} = \frac{1}{45} \begin{pmatrix} 3 & 9 \\ -4 & 3 \end{pmatrix} \neq A^{-1} + B^{-1} = \frac{1}{10} \begin{pmatrix} 2 & 13 \\ -6 & 6 \end{pmatrix}$

(14 b) $A^2 - B^2 = \begin{pmatrix} -11 & -12 \\ 4 & 13 \end{pmatrix} \neq (A - B)(A + B) = \begin{pmatrix} -13 & 24 \\ 20 & 15 \end{pmatrix}$

(15) $(\det A^2) = \det A^t$ (take the "det" of both sides) ; $(\det A)^2 = \det A$ (similar $x^2 = x$)

$(\det A)^2 - \det A = 0$ (factor) ; $\det A (\det A - 1) = 0 \Rightarrow \det A = 0$ or 1

(16) $(kA) \left(\frac{1}{k} A^{-1} \right) = \left(k \cdot \frac{1}{k} \right) A A^{-1} = I$; also $\left(\frac{1}{k} A^{-1} \right) (kA) = I$ so that $(kA)^{-1} = \frac{1}{k} A^{-1}$

" Study " your Inverse theorems \rightarrow this is only one of them!

(17) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; $B = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$; $A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$; $B^{-1} = \frac{1}{-2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$; but $A + B = 0$; $0^{-1} \nexists$

(18) almost any A, B will do! , for example :

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; $B = \begin{pmatrix} -2 & 4 \\ 3 & 1 \end{pmatrix}$, $\det A = -2$, $\det B = -14$

$\det (A + B) = \det \begin{pmatrix} -1 & 6 \\ 6 & 5 \end{pmatrix} = -41 \neq \det A + \det B$