Review Problems # 2

Practice Test # 1

(1) Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ (a) Write A^{-1} as a product of elementary matrices
(b) Write A as a product of elementary matrices

(2) Let $A = \begin{pmatrix} 4 & 2 & 3 \\ 1 & -2 & -4 \\ 3 & -1 & -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ -10 \\ 30 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
(a) find det A , adjoint A and A^{-1} using det A and adjoint A
(b) Solve $A\vec{x} = \vec{b}$ using A^{-1}
(c) Solve for x_3 only using Cramer's Rule

(3) Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$; det $A = -3$ and B equal a 3x3 matrix such that det $B = 5$

find: (a) det $\left(A^{-1}\right)$; (b) det $\left(A^{t}\right)$; (c) det $\left(\frac{1}{3}B\right)$; (d) det $\left(2A^{3}B\right)$

(e) det $\begin{pmatrix} a & b & c \\ 5a+2d & 5b+2e & 5c+2f \\ g-d & h-e & i-f \end{pmatrix}$; (f) det $\begin{pmatrix} a & b & c \\ 3d & 3e & 3f \\ -2d & -2e & -2f \end{pmatrix}$

(4) Find an LU decomposition of $A = \begin{pmatrix} 2 & 6 & 4 \\ 2 & 5 & 8 \\ 3 & 11 & 7 \end{pmatrix}$

(5) Let $A = \begin{pmatrix} 1 & 3 & 15k \\ 2 & 1 & -5 \\ 1 & -2 & 7 \end{pmatrix}$;
(a) find k such that det A = 0 , using (a) find k such that
(b) $A\vec{x} = \vec{O}$ has a parametric solution
(c) A^{-1} $\exists$
(d) $A\vec{x} = \vec{b}$ has a unique solution

(6) If $A^{-1} = A^{t}$, what are the possible values for det A ?

(7) Given the points A (2, -1, 3) , B (4, 0, 2) , C (-2, 2, 1)
(a) find the equation of the line through A and B
(b) determine whether the point (8, 2, 0) lies on the line in (a)
(c) find angle A correct to 2 decimal places ; (d) find the area of triangle ABC

(8) Find, if any, the intersection of the line L_1 and L_2 where

L_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$; L_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 9 \end{pmatrix} + s\begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$

(9) Fill in the blanks in the proof of the following theorem : det $A \neq 0 \Rightarrow A \sim \mathrm{I}$
Proof: Let R = RREF of A , but is $A \sim$ R $\Rightarrow$ _____ A = R $\Rightarrow$ det (____) = det R

det (_____) = ______ $\neq$ 0 (Reason: ______) $\Rightarrow$ det R _____ ; R does not have ______ $\Rightarrow$ R = I

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(10) Evaluate $\det \begin{pmatrix} 3 & -6 & 9 & -3 \\ 0 & 2 & -1 & 7 \\ 2 & -4 & 2 & 2 \\ 3 & 1 & 2 & -1 \end{pmatrix}$

(11) Find K such that (a) the vectors (4, -3, 5) and (k, -6, 10) are parallel

(b) the vectors (4, -3, 5) and (k, -6, 10) are perpendicular.

(12) Prove $\vec{v}\times(2\vec{u}-\vec{v}) = -2(\vec{u}\times\vec{v})$ using cross product properties. (Do not use components)

Answers

(1 a) $A^{-1} = E_3E_2E_1 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is one possible answer.

(1 b) $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(2 a) det A = -5 ; Adj A = $\begin{pmatrix} 0 & 1 & -2 \\ -10 & -17 & 19 \\ 5 & 10 & -10 \end{pmatrix}$; $A^{-1} = -\dfrac{1}{5}\begin{pmatrix} 0 & 1 & -2 \\ -10 & -17 & 19 \\ 5 & 10 & -10 \end{pmatrix}$

(2 b) $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} 14 \\ -138 \\ 75 \end{pmatrix}$; (2 c) $x_3 = \dfrac{\det A_3}{\det A} = \dfrac{-375}{-5} = 75$

(3 a) $\dfrac{1}{\det A} = -\dfrac{1}{3}$; (b) $\det A = -3$; (c) $\left(\dfrac{1}{3}\right)^3 \det B = \dfrac{5}{27}$; (d) $2^3(\det A)^3 \det B = -1080$

(3 e) -6 ; (3 f) 0 (multiple rows)

(4) $L = \begin{pmatrix} 2 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}; U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 9 \end{pmatrix}$ is one possibility

(5 a) $\det A = 0 \Rightarrow k = -\dfrac{4}{5}$; (b) $A\vec{x} = \vec{O}$ has a parametric solution $\Leftrightarrow$ $\det A = 0 \rightarrow k = -\dfrac{4}{5}$;

(c) $A^{-1}\exists \Leftrightarrow$ $\det A \neq 0 \rightarrow k \neq -\dfrac{4}{5}$; (d) $A\vec{x} = \vec{b}$ has a unique solution $\Leftrightarrow$ $\det A \neq 0 \rightarrow k = -\dfrac{4}{5}$

(6) $\det A = \pm 1$

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Answers

(7 a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$; (b) yes! let t = 3 ; (c) $\cos A = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \dfrac{-3}{\sqrt{6}\sqrt{29}} \rightarrow \angle A \approx 103.15^\circ$

or use B   or any multiples   ; (d) Area $\triangle$ ABC = $\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\left\|(1,8,10)\right\| = \dfrac{\sqrt{165}}{2}$

(8) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$

(9) $\underbrace{E_k ....E_1}_{\text{elem. matrices}} A = \mathrm{R} \rightarrow \underbrace{\det E_k ...\det E_1}_{\text{non-zeros (elem. matrices)}} \underbrace{\det A}_{\neq 0 \text{ given}} = \det \mathrm{R} \Rightarrow \det \mathrm{R} \neq 0$

$\Rightarrow$ R does not have a row of zeros $\Rightarrow$ R = I

(10) 624

(11 a) $(4,-3,5) = t(k,-6,10) \Rightarrow t = \frac{1}{2} \Rightarrow k = 8$; (b) $4k + 18 + 50 = 0$ (dot product = 0) $\rightarrow$ k = -17

(12) LHS $= 2(\vec{v} \times \vec{u}) - (\vec{v} \times \vec{v}) = -2(\vec{u} \times \vec{v}) - \vec{O} = -2(\vec{u} \times \vec{v}) =$ RHS

Practice Test # 2

(1) Let $A = \begin{pmatrix} 4 & 4 & -8 \\ 0 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Reduce A to B using 3 elementary row operations.

(b) Find 3 elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = B$

(c) Find 3 elementary matrices F, G, H such that $A = FGHB$

(2) Given $\begin{cases} x_1 - 3x_2 = -5 \\ x_2 + 3x_3 = -1 \\ 2x_1 - 10x_2 + 2x_3 = -20 \end{cases}$; $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}$; $U = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}$

(a) Write the system in the form $A\vec{x} = \vec{b}$

(b) Given that $LU = A$, use L and U (but not A) to solve $A\vec{x} = \vec{b}$

(c) Use Cramer's Rule to solve for x_2 only.

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Practice Test # 2

(3) Let $A=\begin{pmatrix}3 & 0 & -6\\ -4 & -1 & 1\\ 1 & 1 & 2\end{pmatrix};\vec{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix};\vec{b}=\begin{pmatrix}6\\ -3\\ 12\end{pmatrix}$

(a) find an LU decomposition for A

(b) find det A , adj A and A^{-1} using det A and adj A

(c) Solve $A\vec{x}=\vec{b}$ using A^{-1}

(4) Evaluate det $\begin{pmatrix}3 & 1 & -2 & 4\\ 1 & -2 & -3 & 2\\ -2 & 4 & 8 & 3\\ 0 & 2 & -2 & 4\end{pmatrix}$

(5) Let $A=\begin{pmatrix}a & b & c\\ r & s & t\\ x & y & z\end{pmatrix}$ and det $A=2$;

(i) find: (a) det $\left(AA^t\right)$; (b) det $\left(-A^2\right)$; (c) det $\left(3A\right)^{-1}$; (d) det $\begin{pmatrix}a & b & c\\ 3x & 3y & 3z\\ r+2x & s+2y & t+2z\end{pmatrix}$; (e) C_{32}

(ii) How many solutions can $A\vec{x}=\vec{O}$ have ? Why ?

(iii) What is the rank of A ?

(6) Given the points A (1, 2, -4) , B (2, 1, -3) , C (0, 1, 2)

(a) find an equation of the line through A and B

(b) find an equation (in standard form) of the plane containing A , B and C

(c) find a unit vector oppositely directed to $\overrightarrow{BC}$

(d) find the area of triangle ABC

(e) find angle B in triangle ABC to the nearest degree

(f) Does A lie on the line $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}7\\ 8\\ 9\end{pmatrix}+t\begin{pmatrix}6\\ 6\\ 13\end{pmatrix}$? Why or why not ?

(7) Show that L_1: $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}2\\ -3\\ 1\end{pmatrix}+t\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$ and L_2: $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ -2\\ 1\end{pmatrix}+s\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$ are parallel and find the perpendicular distance from L_1 to L_2

(8) Do <u>2</u> out of 3

(a) If A^{-1} exists , prove that det $A \neq 0$;

(b) If $A \sim \mathrm{I}$, prove that both A^{-1} and A can be written as a product of elementary matrices

(c) Prove that $\vec{u}\bullet\left(2\vec{v}\times\vec{w}\right)=\vec{v}\bullet\left(\vec{w}\times 2\vec{u}\right)$

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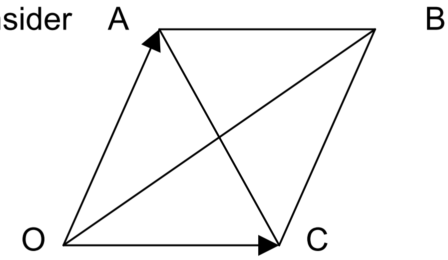
(9) Consider



where $\overrightarrow{OA} = \vec{u}$ and $\overrightarrow{OC} = \vec{v}$

(a) find $\overrightarrow{OB}$ and $\overrightarrow{CA}$ in terms of $\vec{u}$ and $\vec{v}$

(b) simplify the expression $\overrightarrow{OB} \bullet \overrightarrow{CA}$ using $\vec{u}$, $\vec{v}$ and dot product properties

(c) If $\|\vec{u}\| = \|\vec{v}\|$, what is the value of $\overrightarrow{OB} \bullet \overrightarrow{CA}$ in part (b) ?

(d) If $\|\vec{u}\| = \|\vec{v}\|$, what can be said about the angles between the diagonals $\overrightarrow{OB}$ and $\overrightarrow{CA}$?

Bonus : Do the proof that you omitted in # 8

Answers:

(1 a) operations: $\frac{1}{4}R_1 \rightarrow R_1$; $-R_1 + R_3 \rightarrow R_3$; $R_2 \leftrightarrow R_3$

(1 b) $\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{is one possibility}} A = B$; (c) $A = E_1^{-1}E_2^{-1}E_3^{-1}B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B$

(2 a) $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -20 \end{pmatrix}$; (2 b) $A\vec{x} = \vec{b} \Rightarrow L(U\vec{x}) = \vec{b} \Rightarrow L\vec{y} = \vec{b} \Rightarrow \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 2 & -4 & 1 & -20 \end{array}\right) \end{array} \Rightarrow$

$y_1 = -5, y_2 = -1, y_3 = -14 \Rightarrow U\vec{x} = \vec{y} \Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 14 & -14 \end{array}\right) \end{array} \Rightarrow x_1 = 1$, $x_2 = 2$, $x_3 = -1$

(2 c) $x_2 = \dfrac{\det A_2}{\det A} = \dfrac{28}{14} = 2$; (3 a) $L = \begin{pmatrix} 3 & 0 & 0 \\ -4 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & -3 \end{pmatrix}$ is one possibility

(3 b) det $A = 9$, adj $A = \begin{pmatrix} -3 & -6 & -6 \\ 9 & 12 & 21 \\ -3 & -3 & -3 \end{pmatrix}$; $A^{-1} = \dfrac{1}{9}\begin{pmatrix} -3 & -6 & -6 \\ 9 & 12 & 21 \\ -3 & -3 & -3 \end{pmatrix}$; (3 c) $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -8 \\ 30 \\ -5 \end{pmatrix}$; (4) -260

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Answers:

(5 i) (a) $(\text{det } A)(\text{det } A^t) = (\text{det } A)^2 = 4$; (b) $(-1)^3 (\text{det } A)^2 = -4$; (c) $\dfrac{1}{\text{det } (3A)} = \dfrac{1}{3^3 \text{ det } A} = \dfrac{1}{(27)(2)} = \dfrac{1}{54}$

(d) -6 ; (e) $-\text{M}_{32} = -\begin{vmatrix} a & c \\ r & t \end{vmatrix} = -(at - rc)$; (5 ii) $\vec{x} = \vec{O}$ only since det $A \neq 0$; (5 iii) rank $A = 3$ since $\underset{(\text{det } A \neq 0)}{A \sim \text{I}_3}$

(6 a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$; (b) $\vec{n} = k\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = k(-5, -7, -2)$; Let $\vec{n} = (5, 7, 2) \rightarrow (5, 7, 2) \bullet (x - 1, y - 2, z + 4) = 0$

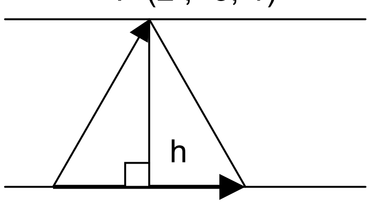
$\rightarrow 5x + 7y + 2z - 11 = 0$ (c) $\dfrac{-1}{\sqrt{29}}(-2, 0, 5)$; (d) Area $= \dfrac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \dfrac{1}{2}\left\|(-5, -7, -2)\right\| = \dfrac{1}{2}\sqrt{78}$ square units

(e) $\cos B = \dfrac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left\|\overrightarrow{BA}\right\| \left\|\overrightarrow{BC}\right\|} = \dfrac{(-1, 1, -1) \bullet (-2, 0, 5)}{\sqrt{3}\ \sqrt{29}} = \dfrac{-3}{\sqrt{87}} \rightarrow \angle B \approx 109^\circ$; (e) yes! $t = -1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 6 \\ 6 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

(7) $\vec{d_1} = \vec{d_2} = \begin{pmatrix} -1 & 1 & 2 \end{pmatrix}$ since L_1 and L_2 do not intersect and $\vec{d_1} = \vec{d_2}$, L_1 parallel to L_2

$\rightarrow \begin{cases} 2 - t = -s \\ -3 + t = -2 + s \Rightarrow \\ 1 + 2t = 1 + 2s \end{cases} \quad \begin{matrix} -t + s = -2 \\ t - s = 1 \\ 2t - 2s = 0 \end{matrix} \Rightarrow \underbrace{\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{array}\right]}_{\text{Inconsistent}}$; $\frac{1}{2}$bh = $\dfrac{1}{2}\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\| \rightarrow$ h $= \dfrac{\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|} = \dfrac{\left\|(2, -1, 0) \times (-1, 1, 2)\right\|}{\left\|(-1, 1, 2)\right\|}$

P (2 , -3, 1)



A (0, -2, 1) B (-1, -1, 3)

h $= \dfrac{\left\|(-2, -4, 1)\right\|}{\left\|(-1, 1, 2)\right\|} = \dfrac{\sqrt{21}}{\sqrt{6}} = \sqrt{\dfrac{7}{2}} \approx 1.87$

(8 a) $AA^{-1} = \text{I} \Rightarrow \text{det}(AA^{-1}) = \text{det I} \Rightarrow \text{det}(A) \bullet \Rightarrow \text{det}(A^{-1}) = 1$

$\Rightarrow \text{det}(A) \neq 0$

(8b) $A \sim \text{I} \Rightarrow E_k \ldots . E_2 E_1 A = \text{I} \Rightarrow A^{-1} = E_k \ldots . E_2 E_1$

$A = E_2^{-1} E_1^{-1} \ldots E_k^{-1}$; (8 c) LHS $\begin{vmatrix} \vec{u} \\ 2\vec{v} \\ \vec{w} \end{vmatrix} = 2 \underset{\text{R}_1 \leftrightarrow \text{R}_2}{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} = -2 \underset{\text{R}_2 \leftrightarrow \text{R}_3}{\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ 2u_1 & 2u_2 & 2u_3 \end{vmatrix} =$ RHS

(9 a) $\overrightarrow{OB} = \vec{u} + \vec{v}, \overrightarrow{CA} = \vec{u} - \vec{v}$; (9 b) $(\vec{u} + \vec{v}) \bullet (\vec{u} - \vec{v}) = \vec{u} \bullet \vec{u} + \vec{v} \bullet \vec{u} - \vec{u} \bullet \vec{v} - \vec{v} \bullet \vec{v}$

$= \vec{u} \bullet \vec{u} + \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{v} - \vec{v} \bullet \vec{v} = \left\|\vec{u}\right\|^2 + \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{v} - \left\|\vec{v}\right\|^2 = \left\|\vec{u}\right\|^2 - \left\|\vec{v}\right\|^2$;

(9 c) $(\vec{u} + \vec{v}) \bullet (\vec{u} - \vec{v}) = 0$ All angles are 90° (where diagonals meet)