MATRIX SUBSPACES

For each of the following (a) find a basis for the column space of A, (b) find a basis for the null space of A (solution space), (c) find a basis for the row space of A, (d) state the dimension for each of the spaces in (a), (b), (c) and describe each of the spaces in (a), (b), (c), (e) write dependency equations (linear combinations) for each of the nonbasic column vectors of A in terms of the basic column vectors of A

1. $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \\ -2 & 5 & 0 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Note: $\det(A) = -51 \neq 0 \Rightarrow A \sim R = I$

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & -2 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Note: $\det(A) = 0 \Rightarrow R \neq I$ (R has a row of zeroes)

4. $A = \begin{bmatrix} 1 & 2 & 3 & 16 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 2 & 3 & 16 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Let R = the RREF of A

Answers: In all questions, assume $\vec{a_1}$ = first column vector of A ; $\vec{a_2}$ = second column vector of A, etc

(1) (a) $\left\{\vec{a_1}, \vec{a_2}, \vec{a_3}\right\}$; (b) none ; (c) $\left\{(1,0,-2),(3,4,1),(-2,5,0)\right\}$ or $\left\{(1,0,0),(0,1,0),(0,0,1)\right\}$

(d) d(Col A) = 3 = dim (Row A) ; dim (null A) = 0 ; Colsp A = RowSp A = all of $\mathbb{R}^3$;

Null A = $\vec{O}$ in $\mathbb{R}^3$; (e) no dependency equations (col. vectors of A are LI)

(2) (a) $\left\{\vec{a_1}, \vec{a_3}\right\}$; (b) $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$; (c) $\left\{(1,2,3),(3,6,-2)\right\}$ or $\left\{(1,2,0),(0,0,1)\right\}$

(d) dim (Col A) = 2 = dim (Row A) ; dim (Null A) = 1 ; Col A = $\mathbb{R}^2$; Row A =

a plane in $\mathbb{R}^3$ spanned by $(1,2,0)$ and $(0,0,1)$; Null A = a line in $\mathbb{R}^3$ with

$\vec{d} = (-2,1,0)$; (e) $\vec{a_2} = 2\vec{a_1} + 0\vec{a_3}$

Matrix Subspaces

(3) (a) $\left\{\overrightarrow{a_1},\overrightarrow{a_2}\right\}$; (b) $\left\{\left(1,-2,1\right)\right\}$; (c) $\left\{\left(1,2,3\right),\left(4,5,6\right)\right\}$ or $\left\{\left(1,0,-1\right),\left(0,1,2\right)\right\}$

(d) dim(Col A) = 2 = dim (Row A) ; dim (null A) = 1 ; Col A = a plane in $\mathbb{R}^3$ spanned by $\left(1,4,7\right),\left(2,5,8\right)$;

Null A = a line in $\mathbb{R}^3$ with $\vec{d}=\left(1,-2,1\right)$; Row A = a plane in $\mathbb{R}^3$ spanned by $\left(1,0,-1\right),\left(0,1,2\right)$;

(e) $\overrightarrow{a_3}=-\overrightarrow{a_1}+2\overrightarrow{a_2}$

(4) (a) $\left\{\overrightarrow{a_1},\overrightarrow{a_2},\overrightarrow{a_4}\right\}$; (b) $\left\{\left(1,-2,1,0\right)\right\}$; (c) the 3 row vectors of A or of R

(d) dim (Col A) = 3 = dim (Row A) ; dim (Null A) = 1 ; Col A = $\mathbb{R}^3$;

Row A = a 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\left(1,0,-1,0\right)$, $\left(0,1,2,0\right)$, $\left(0,0,0,1\right)$;

Null A = a 1-dimensional subspace of $\mathbb{R}^4$ spanned by $\left(1,-2,1,0\right)$; (e) $\overrightarrow{a_3}=-\overrightarrow{a_1}+2\overrightarrow{a_2}+0\overrightarrow{a_4}$

(5) (a) $\left\{\overrightarrow{a_1},\overrightarrow{a_2},\overrightarrow{a_4}\right\}$; (b) $\left\{\left(1,-2,1,0\right)\right\}$; (c) $\left\{\left(1,2,3,16\right),\left(4,5,6,15\right),\left(7,8,9,24\right)\right\}$ or

$\left\{\left(1,0,-1,0\right),\left(0,1,2,0\right),\left(0,0,0,1\right)\right\}$; (d) dim (Col A) = 3 = dim (Row A) ;

dim (Null A) = 1 ; Col A = a 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\overrightarrow{a_1},\overrightarrow{a_2},\overrightarrow{a_4}$;

Row A = a 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\left(1,0,-1,0\right)$, $\left(0,1,2,0\right)$, $\left(0,0,0,1\right)$;

Null A = a 1-dimensional subspace of $\mathbb{R}^4$ spanned by $\left(1,-2,1,0\right)$; (e) $\overrightarrow{a_3}=-\overrightarrow{a_1}+2\overrightarrow{a_2}+0\overrightarrow{a_4}$

(6) (a) none ; (b) $\left\{\left(1,0,0\right),\left(0,1,0\right),\left(0,0,1\right)\right\}$; (c) none ; (d) dim (Col A) = 0 = dim (Row A) ;

dim (Null A) = 3 ; Col A = $\vec{O}$ in $\mathbb{R}^4$; Null A = $\mathbb{R}^3$, Row A = $\vec{O}$ in $\mathbb{R}^3$; (e) not applicable

(7) (a) $\left\{\overrightarrow{a_1},\overrightarrow{a_3},\overrightarrow{a_6}\right\}$; (b) $\left\{\left(-3,1,0,0,0,0\right),\left(-4,0,-2,1,0,0\right),\left(-2,0,0,0,1,0\right)\right\}$;

(c) $\left\{\left(1,3,0,4,2,0\right),\left(0,0,1,2,0,0\right),\left(0,0,0,0,0,1\right)\right\}$; (d) dim (Col A) = dim (Row A) = 3 ;

dim (Null A) = 3 ; Col A = a 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\overrightarrow{a_1},\overrightarrow{a_3},\overrightarrow{a_6}$;

Null A = a 3-dimensional subspace of $\mathbb{R}^6$ spanned by $\left(-3,1,0,0,0,0\right),\left(-4,0,-2,1,0,0\right),\left(-2,0,0,0,1,0\right)$;

Row (A) = a 3-dimensional subspace of $\mathbb{R}^6$ spanned by $\left(1,3,0,4,2,0\right),\left(0,0,1,2,0,0\right),\left(0,0,0,0,0,1\right)$;

(e) $\overrightarrow{a_2}=3\overrightarrow{a_1}$, $\overrightarrow{a_4}=4\overrightarrow{a_1}+2\overrightarrow{a_3}$, $\overrightarrow{a_5}=2\overrightarrow{a_1}$

(8) (a) $\left\{\overrightarrow{a_1},\overrightarrow{a_3},\overrightarrow{a_4}\right\}$; (b) $\left\{\left(-1,1,0,0,0\right),\left(-1,0,-1,0,1\right)\right\}$;

(c) $\left\{\left(1,1,0,0,1\right),\left(0,0,1,0,1\right),\left(0,0,0,1,0\right)\right\}$; (d) dim (Col A) = dim (Row A) = 3 ;

dim (Null A) = 2 ; Col A = a 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\overrightarrow{a_1},\overrightarrow{a_3},\overrightarrow{a_4}$;

Row (A) = a 3-dimensional subspace of $\mathbb{R}^5$ spanned by $\left(1,1,0,0,1\right),\left(0,0,1,0,1\right),\left(0,0,0,1,0\right)$;

Null A = a 2-dimensional subspace of $\mathbb{R}^5$ spanned by $\left(-1,1,0,0,0\right),\left(-1,0,-1,0,1\right)$