Inverses Exercise 1

(1) (a) Assume $A^{-1} \exists$, solve $BA = C$ for B (square matrices)

(b) Assume $A^{-1}, B^{-1} \exists$ and $ACB = ADB$, prove $C = D$ (square matrices)

(c) Solve $B = P^{-1} A P$ for A (square matrices)

(2) Assume A, B, C, D are square matrices and a is a scalar.
State assumptions that allow one to solve for B , then solve for B.

(a) $AB + aB = D$; (b) $AB + 2C = DB$; (c) $AB = C + 3B$
(d) $BA + BC = C$; (e) $BA = CA$

(3) Assume A , B , C , D are square and invertible. Find the inverse of
(a) $ABCD$; (b) $ABA^{-1}B^{-1}$

(4) Assume $A^{-1} \exists$, find the inverse of (a) A^3 ; (b) $3A$; (c) $-A^4$; (d) $\left(A^{-1}\right)^2$

(5) Find A^{-1} if (a) $A^2 - 2A - 3I = 0$; (b) $A^6 - 5A^4 + 3A^2 + 2I = 0$

(6) (a) Assume $B \neq 0$ and $AB = 0$, can $A^{-1} \exists$?

(b) Assume $A^2 = 0$ and $A \neq 0$, can $A^{-1} \exists$?
What if $A = 0$? Does A^{-1} exist in that case ?

(c) Assume $P^2 = P$ and $P \neq I$, can $P^{-1} \exists$?

Answers :

(1a) CA^{-1} ; (1c) PBP^{-1} ; (2a) $\left(A + aI\right)^{-1} D$ where $(A + aI)^{-1}$ must $\exists$;

(2b) $(-2)(A - D)^{-1} C$ or $2(D - A)^{-1} C$ where $(A - D)^{-1}$ must $\exists$;

(2c) $(A - 3I)^{-1} C$ where $(A - 3I)^{-1}$ must $\exists$; (2d) $C(A + C)^{-1}$ where $(A + C)^{-1}$ must $\exists$;

(2e) C where $A^{-1} \exists$

(3a) $D^{-1}C^{-1}B^{-1}A^{-1}$; (3b) $BAB^{-1}A^{-1}$

(4a) $(A^3)^{-1} = (A^{-1})^3 = A^{-3}$; (4b) $\frac{1}{3}A^{-1}$; (4c) $-(A^{-1})^4 = -(A^4)^{-1} = -A^{-4}$; (4d) A^2

(5a) $\frac{1}{3}(A - 2I)$; (5b) $\frac{1}{2}(5A^3 - A^5 - 3A)$

Inverses Exercise 2

(1) Let $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(a) Verify that $(AB)^{-1} = B^{-1}A^{-1}$; (b) Verify that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(2) Find A given that

(a) $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$; (b) $(7A)^{-1} = \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}$; (c) $(5A^{t})^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$;

(d) $(I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$

(3) Let $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$, show that $(A^{-1})^{3} = (A^{3})^{-1}$

(4) (a) Given $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, compute A^{-1} . (b) Verify that $AA^{-1} = I$

(c) Write the system $\begin{cases} 2x + 3y + 4z = -1 \\ 4x + 3y + z = 5 \\ x + 2y + 4z = -2 \end{cases}$ in the form $AX = B$ or $A\vec{x} = \vec{b}$ and use A^{-1} to solve the system.

(c) Write the system $\begin{cases} 2x + 3y + 4z = 1 \\ 4x + 3y + z = 2 \\ x + 2y + 4z = 0 \end{cases}$ in the form $AX = B$ or $A\vec{x} = \vec{b}$ and use A^{-1} to solve the system.

(c) Write the system $\begin{cases} 2x + 3y + 4z = 0 \\ 4x + 3y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$ in the form $AX = B$ or $A\vec{x} = \vec{b}$ and use A^{-1} to solve the system.

(f) If $B = \begin{pmatrix} 8 & 12 & 16 \\ 16 & 12 & 4 \\ 4 & 8 & 16 \end{pmatrix}$, write $B = kA$ (k = a scalar) and

$B^{-1} = (kA)^{-1}$, that is find B^{-1} using A^{-1}

(g) If $C = \begin{pmatrix} -10/3 & 4/3 & 3 \\ 5 & -4/3 & -14/3 \\ -5/3 & 1/3 & 2 \end{pmatrix}$, write $C = kA$ (k = a scalar),

$C = kA^{-1}$ and $C^{-1} = (kA^{-1})^{-1}$, that is find C^{-1} using A^{-1}

Inverses Exercise 2

(5) Given $A^{-1} = \begin{pmatrix} 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 \\ -2/5 & 1/10 & 1/10 \end{pmatrix}$, find A .

(6) Given $A = \begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix}$, verify that $A^2 = A$

(7) Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (a) If $BC = CB$, show that B is of form $\begin{bmatrix} t & s \\ s & t \end{bmatrix}$;

(b) If $CB = 0$, show that B is of form $\begin{bmatrix} s & t \\ -s & -t \end{bmatrix}$

(8) Find 4 matrices satisfying $A^2 = I_3$ but $A \neq I_3$

(9) Assume A , B , X are square matrices of the same size , $A^{-1} \exists , B^{-1} \exists$; solve for X .

(a) $AX = AB$; (b) $AX = BA$; (c) $AX = A + B$; (d) $ABX = A + B$;

(e) $AXB = A + B$; (f) $XAB = A + B$

Answers :

(2 a) $A = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}$; (2 b) $A = \frac{1}{7}\begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$; (2 c) $A = \frac{1}{5}\begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$; (2 d) $\frac{1}{13}\begin{pmatrix} -9 & 1 \\ 2 & -6 \end{pmatrix}$

(4 a) $A^{-1} = \frac{1}{5}\begin{pmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{pmatrix}$; (4 c) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12/5 \\ -7/5 \\ -2/5 \end{pmatrix}$;

(4 d) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/5 \\ 7/5 \\ -3/5 \end{pmatrix}$; (4 e) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (trivial solution only)

(4 f) $B = 4A \Rightarrow B^{-1} = (4A)^{-1} = \frac{1}{4}A^{-1} = \frac{1}{20}\begin{pmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{pmatrix}$; (4 g) $C = \frac{5}{3}A^{-1} \Rightarrow C^{-1} = \left(\frac{5}{3}A^{-1}\right)^{-1} = \frac{3}{5}A$

(5) $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$; (8) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, etc .

(9 a) $X = B$; (b) $X = A^{-1}BA$; (c) $X = I + A^{-1}B$; (d) $X = B^{-1} + B^{-1}A^{-1}B$ or $B^{-1} + (AB)^{-1}B$

(9 e) $X = B^{-1} + A^{-1}$; (9 f) $X = AB^{-1}A^{-1} + A^{-1}$ or $A(AB)^{-1} + A^{-1}$