Math - Calculus II

SERIES SUMMARY

Geometric Series: Series of form $a + ar + ar^2 + \dots.$

geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$ and divergent if $|r| \geq 1$

Examples:

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \frac{5}{2} + \frac{5}{2^2} + \frac{5}{2^3} + \dots \quad \left(a = \frac{5}{2} , r = \frac{1}{2} \right) \quad \text{convergent G.S.}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{e^{n-1}} = -4 + \frac{4}{e} - \frac{4}{e^2} + \dots \quad \left(a = -4 , r = -\frac{1}{e} \right) \quad \text{convergent G.S.}$$

$$\sum_{n=1}^{\infty} 4\left(\frac{5}{3}\right)^n = 4\left(\frac{5}{3}\right) + 4\left(\frac{5}{2}\right)^2 + \dots \quad \left(a = \frac{20}{3} , r = \frac{5}{3} \right) \quad \text{divergent G.S.}$$

$$S_n \quad \text{for a G.S. is} \quad \frac{a(1-r^n)}{1-r} \rightarrow \frac{a}{1-r} \quad \text{as} \quad n \rightarrow \infty \quad \text{if} \quad |r| < 1$$

P-Series: Series of form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$:

if $0 < p \leq 1$, the p-series diverges ; if $p > 1$, the p-series converges

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad (\text{divergent}) \quad ; \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad (\text{divergent})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{convergent}) \quad ; \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad (\text{convergent})$$

TESTS

(1) nth term test (N.T.T.)

$$\lim_{n\to\infty} a_n \neq 0 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges} \;;\; \text{if } \lim_{n\to\infty} a_n = 0 \;,\; \text{test fails}$$

(2) Ratio test (RatioT) (only works if a_n contains an exponential or factorial)

$$\text{Given } \sum_{n=1}^{\infty} a_n \;,\; \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \begin{cases} \text{if } 0 \le L < 1 \rightarrow \text{convergent series} \\ \text{if } L > 1 \rightarrow \text{divergent series} \\ \text{if } L = 1 \rightarrow \text{test fails} \end{cases}$$

(3) Integral test (I.T.) (used only for positive term series)

conditions: $f(x) \ge 0$ and continuous for $x \ge 1$, $f(x)$ decreasing

$$\int_1^{\infty} f(x)\,dx = \infty \text{ (diverges)} \rightarrow \text{series } \sum a_n \text{ diverges}$$

$$\int_1^{\infty} f(x)\,dx = L \text{ (converges)} \rightarrow \text{series } \sum a_n \text{ converges}$$

(4) Comparison tests (used only for positive term series)

Direct Comparison test (D.C.T.)

Let $\sum_{n=1}^{\infty} a_n$ be the series being tested ; $\sum_{n=1}^{\infty} b_n$ is a series selected normally a p-series or a geometric series.

(a) $a_n \le k\, b_n$; if the larger $k \sum b_n$ converges , then the smaller $\sum a_n$ converges

(b) $k\, b_n \le a_n$; if the smaller $k \sum b_n$ diverges , then the larger $\sum a_n$ diverges

Limit Comparison test (L.C.T.)

(a) $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$

then both series behave the same way; that is, both converge or both diverge.

(b) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then both series diverge (the selected series $\sum_{n=1}^{\infty} b_n$ must be divergent)

(b) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then both series converge (the selected series $\sum_{n=1}^{\infty} b_n$ must be convergent)

(5) Root test (RootT) (only applies to series of form $\sum_{n=1}^{\infty} \left(\quad \right)^n$)

$$\text{Given } \sum_{n=1}^{\infty} a_n \ , \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \begin{cases} \text{if } 0 \le L < 1 \rightarrow \text{convergent series} \\ \text{if } L > 1 \rightarrow \text{divergent series} \\ \text{if } L = 1 \rightarrow \text{test fails} \end{cases}$$

(6) Absolute convergence implies convergence
If the corresponding series of positive terms converges, then the given series converges.
The series of positive terms is the largest number of the "family" of series.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (p-series), then $\sum_{n=1}^{\infty} \frac{-1}{n^2}$ converges (all terms are negative). Also $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges (alternating series).

(7) Alternating series test (A.S.T.) (applies to alternating series , signs alternate)

$a_1 - a_2 + a_3 - a_4 + \dots$ or $-a_1 + a_2 - a_3 + a_4 - \dots$

conditions: if $\lim_{n\to\infty} a_n = 0$ and a_n decreasing

then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n}$ decreasing

If the sum of an alternating series is approximated by the sum of the firest n terms, then the remainder is less than $|a_{n+1}|$

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$

if we approximate the sum of the infinite series by S_6 , then the sum of the rest of the terms from $\frac{1}{7}$ onward $< \frac{1}{7}$; remainder $= \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$

$$\text{remainder} = \frac{1}{7} - \left(\frac{1}{8} - \frac{1}{9}\right) - \left(\frac{1}{10} - \frac{1}{11}\right) - \left(\frac{1}{12} - \frac{1}{13}\right) - \dots$$

$$= \frac{1}{7} - \frac{1}{72} - \frac{1}{110} - \frac{1}{156} - \dots < \frac{1}{7}$$

(8) If a series consists of <u>only</u> <u>negative</u> terms such as $\sum_{n=1}^{\infty} \frac{-1}{n}$, for example, we pull out (-1) and test the positive term series for convergence or divergence.

$\sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} = (\text{ constant }) \sum_{n=1}^{\infty} \frac{1}{n}$; since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series) then

$(-1) \sum_{n=1}^{\infty} \frac{1}{n}$ is also divergent.