Math - Calculus II <u>Ratio Test (RatioT) ; Root Test (RootT)</u>

Determine whether the series converges (absolutely) or diverges :

(1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 3^n}{n!}$ convergent (absolutely) by RatioT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$

(2) $\sum_{n=1}^{\infty} \frac{4^n}{n^2}$ divergent by RatioT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 4 > 1$

(3) $\sum_{k=1}^{\infty} k \left(-\frac{1}{2} \right)^k$ convergent (absolutely) by RatioT: $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{2} < 1$

(4) $\sum_{k=1}^{\infty} k^k$ divergent by RootT: $\lim_{k\to\infty} \sqrt[k]{|\, a_k \,|} = \infty > 1$

(5) $\sum_{n=1}^{\infty} \frac{(n!)^2\, 2^n}{(2n+1)!}$ convergent (absolutely) by RatioT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1$

(6) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n+2}{2n-1} \right)^n$ divergent by RootT: $\lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} = \frac{3}{2} > 1$

(7) $\sum_{n=1}^{\infty} \left(\frac{4}{2n-1} \right)^n$ convergent (absolutely) by RootT: $\lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} = 0 < 1$

(8) $\sum_{m=1}^{\infty} (-1)^{m+1} \left(\frac{m+2}{3m-1} \right)^m$

convergent (absolutely) by RootT: $\lim_{m\to\infty} \sqrt[m]{|\, a_m \,|} = \left(\frac{1}{3} \right)^2 < 1$

(9) $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^3}{e^n}$ convergent (absolutely) by RatioT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{e} < 1$

(10) $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^n}{n!}$ divergent by RatioT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = e > 1$

(11) $\displaystyle\sum_{n=1}^{\infty} \left(1 + e^{-n} \right)^n$ $\left\{ \begin{array}{l} \text{RootT fails since } \displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \\ \text{series diverges ny N.T.T.: } \displaystyle\lim_{n\to\infty} a_n = 1 \neq 0 \end{array} \right.$

not an easy limit, use logs & LH

Classify the series as conditionally convergent, absolutely convergent or divergent.
Specify the test(s) used and give all the "details"!

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n}$

Conditionally convergent : $\sum \frac{(-1)^{n+1}}{3n}$ convergent by A.S.T. ; $\frac{1}{3} \sum \frac{1}{n}$ divergent by p-series

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

Absolutely convergent : $\sum \frac{(-1)^{n+1}}{n^{4/3}}$ convergent by A.S.T. ; $\sum \frac{1}{n^{4/3}}$ convergent by p-series

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2}$ divergent by RatioT: $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 4 > 1$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!}$ convergent (absolutely) by RatioT: $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Conditionally convergent : $\sum \frac{(-1)^n}{n}$ convergent by A.S.T. ; $\sum \frac{1}{n}$ divergent by p-series

(6) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$

Conditionally convergent : $\sum (-1)^n \frac{\ln(n)}{n}$ convergent by A.S.T. ; $\sum \frac{\ln(n)}{n}$ divergent by I.T.

(7) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+2}{3n+1}\right)^n$ convergent (absolutely) by RootT: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{3} < 1$

(8) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$

$$\begin{cases} \text{Absolutely convergent : } \sum \frac{(-1)^{n+1}}{n^2+1} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \frac{1}{n^2+1} \le \frac{1}{n^2} \text{ for } n \ge 1 \\ \text{larger } \sum \frac{1}{n^2} \text{ convergent} \rightarrow \text{smaller series } \sum \frac{1}{n^2+1} \text{ convergent} \end{cases}$$

(9) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+2}{n^3+3n}$

$$\begin{cases} \text{Absolutely convergent : } \sum (-1)^{n+1} \frac{n+2}{n^3+3n} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \frac{n+2}{n^3+3n} \le \frac{3n}{n^3} = \frac{3}{n^2} \text{ for } n \ge 1 \\ \text{larger } 3\sum \frac{1}{n^2} \text{ convergent} \rightarrow \text{smaller series } \sum \frac{n+2}{n^3+3n} \text{ convergent} \end{cases}$$

(10) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

$$\begin{cases} \text{Conditionally convergent : } \sum (-1)^{n+1} \frac{n^2}{n^3+1} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \frac{n^2}{2n^3} = \frac{1}{2n} \le \frac{n^2}{n^3+1} \text{ for } n \ge 1 \\ \text{smaller } \frac{1}{2}\sum \frac{1}{n} \text{ divergent} \rightarrow \text{larger series } \sum \frac{n^2}{n^3+1} \text{ divergent} \end{cases}$$

(11) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1}$ divergent by N.T.T.: $\lim_{n\to\infty} |a_n| = 1 \neq 0$

(12) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$

$$\begin{cases} \text{Absolutely convergent : D.C.T.: } \frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3} \text{ for } n \geq 1 \\ \text{larger } \sum \frac{1}{n^3} \text{ convergent} \rightarrow \text{smaller series } \sum \frac{|\sin(n)|}{n^3} \text{ convergent absolutely} \end{cases}$$

(13) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$

Conditionally convergent : $\sum \frac{(-1)^n}{n \ln(n)}$ convergent by A.S.T. ; I.T.: $\sum \frac{1}{n \ln(n)}$ divergent

(14) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n}}$

$$\begin{cases} \text{Conditionally convergent : } \sum \frac{(-1)^n}{\sqrt{n^2+n}} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \frac{1}{3n} \leq \frac{1}{\sqrt{n^2+n}} \text{ for } n \geq 1 \\ \text{smaller } \frac{1}{3} \sum \frac{1}{n} \text{ divergent} \rightarrow \text{larger series } \sum \frac{1}{\sqrt{n^2+n}} \text{ divergent} \end{cases}$$

(15) $\sum_{n=2}^{\infty} \left(\frac{-1}{\ln(n)}\right)^n$ convergent (absolutely) by RootT: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$

(16) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1} + \sqrt{n}}$

$$\begin{cases} \text{Conditionally convergent : } \sum \frac{(-1)^n}{\sqrt{n+1} + \sqrt{n}} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \frac{1}{3\sqrt{n}} \leq \frac{1}{\sqrt{n+1} + \sqrt{n}} \text{ for } n \geq 1 \\ \text{smaller } \frac{1}{3} \sum \frac{1}{\sqrt{n}} \text{ divergent} \rightarrow \text{larger series } \sum \frac{1}{\sqrt{n+1} + \sqrt{n}} \text{ divergent} \end{cases}$$

(17) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (n^2+1)}{n^3+2}$

$$\begin{cases} \text{Conditionally convergent : } \sum \dfrac{(-1)^n\ (n^2+1)}{n^3+2} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \dfrac{1}{3n} = \dfrac{n^2}{3n^3} \le \dfrac{n^2+1}{n^3+2} \text{ for } n \ge 1 \\ \text{smaller } \dfrac{1}{3} \sum \dfrac{1}{n} \text{ divergent} \rightarrow \text{larger series } \sum \dfrac{n^2+1}{n^3+2} \text{ divergent} \end{cases}$$

(18) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ \cos(n\pi)}{n^4+1} = \sum_{n=1}^{\infty} \frac{(-1)^n\ n}{n^4+1}$

$$\begin{cases} \text{Absolutely convergent : } \sum \dfrac{(-1)^n\ n}{n^4+1} \text{ convergent by A.S.T.} \\ \text{D.C.T.: } \dfrac{n}{n^4+1} \le \dfrac{n}{n^4} = \dfrac{1}{n^3} \text{ for } n \ge 1 \\ \text{larger } \sum \dfrac{1}{n^3} \text{ convergent} \rightarrow \text{smaller series } \sum \dfrac{n}{n^4+1} \text{ convergent} \end{cases}$$