Math - Calculus II POWER SERIES
Sample problems

(1) $\sum_{n=1}^{\infty} n!\ x^n$; ratio test: $\lim_{n\to\infty} \left| \frac{(n+1)!\ x^{n+1}}{n!\ x^n} \right| = \infty > 1$; divergent for all x except when x = 0 ; center = 0 ; radius = 0 ; interval of convergence: { 0 } ; convergent to 0

(2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ (x+2)^n}{n!}$; ratio test: $\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x+2)^n} \right| = 0 < 1$ for all x

center = -2 ; radius = ∞ ; interval of convergence = $(-\infty\ ,\ \infty)$

Note: Examples 1 and 2 illustrate “extreme” cases !
presence of factorial in numerator implies that the power series converges only at the center
presence of factorial in denominator implies that the power series converges everywhere

(3) $\sum_{n=0}^{\infty} \frac{(-1)^n\ (2x+3)^n}{5^n\ (4n+3)^2}$;

ratio test: $\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{5^{n+1}\ (4n+7)^2} \cdot \frac{5^n\ (4n+3)^2}{(2x+3)^n} \right| = \frac{|2x+3|}{5} < 1$

$|2x+3| < 5 \rightarrow -4 < x < 1$ **series converges and diverges** $x > 1$ **or** $x < -4$

ratio test fails when x = 1 and -4

at x = 1 $\rightarrow$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)^2}$ converges since $\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2}$ is absolutely convergent

at x = -4 $\rightarrow$ $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(4n+3)^2} = \sum_{n=0}^{\infty} \frac{1}{(4n+3)^2}$; C.T. with $\sum \frac{1}{n^2}$ (conv)

both series converge

center = $-\frac{3}{2}$; radius = $\frac{5}{2}$; interval of convergence = [-4 , 1]

(4) $\sum_{n=2}^{\infty} \frac{(-1)^n (x-3)^n}{n \ln n}$;

ratio test: $\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln(n)}{(x-3)^n} \right| = |x-3| < 1$

$|x-3| < 1 \rightarrow 2 < x < 4$ series converges and diverges $x > 4$ or $x < 2$

ratio test fails when x = 2 and 4

at x = 2 $\rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by integral test

at x = 4 $\rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by A.S.T.

center = 3 ; radius = 1 ; interval of convergence =]2, 4]

(5) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n (x-1)^n}{3n+2}$;

ratio test: $\lim_{n\to\infty} \left| \frac{4^{n+1} (x-1)^{n+1}}{3n+5} \cdot \frac{3n+2}{4^n (x-1)^n} \right| = 4 |x-1| < 1$

$4|x-1| < 1 \rightarrow \frac{3}{4} < x < \frac{5}{4}$ series converges and diverges $x > \frac{5}{4}$ or $x < \frac{3}{4}$

ratio test fails when $x = \frac{3}{4}$ and $\frac{5}{4}$

at $x = \frac{3}{4} \rightarrow -\sum_{n=1}^{\infty} \frac{1}{3n+2}$; compare with $\sum \frac{1}{n}$ (div)

at $x = \frac{5}{4} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+2}$ converges by A.S.T.

center = $x = \frac{1}{4}$; radius = 1 ; interval of convergence = $\left(\frac{3}{4}, \frac{5}{4} \right]$

(6) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\ (x+1)^n}{(\ln\ n)(3^n)}$;

ratio test: $\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{\ln(n+1)\ 3^{n+1}} \cdot \frac{3^n\ \ln\ n}{(x+1)^n} \right| = \frac{1}{3}\ |x+1| < 1$

$|x+1| < 3$ ➜ $-4 < x < 2$ series converges and diverges $x > 2$ or $x < -4$

ratio test fails when $x = 2$ and -4

at $x = 2$ ➜ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln\ n}$; AST (conv)

at $x = -4$ ➜ $-\sum_{n=1}^{\infty} \frac{1}{\ln\ n}$ (div)

center : x = -1 ; radius = 3 ; interval of convergence = $(\ -4\ ,\ 2\ ]$

Interval Notation: $[\ a\ ,\ b\ ]$ ➜ $a \le x \le b$; $(\ a\ ,\ b\ )$ ➜ $a < x < b$

$[\ a\ ,\ b\ )$ ➜ $a \le x < b$; $(\ a\ ,\ b\ ]$ ➜ $a < x \le b$