Math - Calculus II

Direct Comparison Test (DCT) ; Limit Comparison Test (LCT)

Determine convergence or divergence :

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$

D.C.T. : $\sum \frac{1}{\sqrt{n}}$ div p−series $p = \frac{1}{2}$

$\frac{1}{3\sqrt{n}} \le \frac{1}{\sqrt{2n+1}}$ for $n \ge 1$; smaller $\frac{1}{3} \sum \frac{1}{\sqrt{n}}$ div ➜ larger series $\sum \frac{1}{\sqrt{2n+1}}$ div

L.C.T.: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \frac{1}{\sqrt{2}} > 0$ ➜ both series $\sum \frac{1}{\sqrt{n}}$, $\sum \frac{1}{\sqrt{2n+1}}$ diverges

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ 2^n}$

D.C.T. : $\sum \frac{1}{2^n}$ conv geo−series , $r = \frac{1}{2}$

$\frac{1}{n\ 2^n} \le \frac{1}{2^n}$ for $n \ge 1$; larger $\sum \frac{1}{2^n}$ conv ➜ smaller series $\sum \frac{1}{n\ 2^n}$ conv

L.C.T.: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n\ 2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{n} = 0$ ➜ both series $\sum \frac{1}{2^n}$, $\sum \frac{1}{n\ 2^n}$ converges

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^2}$

D.C.T. : $\sum \frac{1}{n^2}$ conv p−series $p = 2$

$\frac{2+\sin(n)}{n^2} \le \frac{3}{n^2}$ for $n \ge 1$ Note: $2+\sin(n) \ge 0$ since $-1 \le \sin(n) \le 1$

larger $3 \sum \frac{1}{n^2}$ conv ➜ smaller series $\sum \frac{2+\sin(n)}{n^2}$ conv

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^3}$

D.C.T. : $\sum \frac{1}{n^3}$ conv p−series p = 3

$\frac{|\cos(n)|}{n^3} \leq \frac{1}{n^3}$ for $n \geq 1$ Note: $|\cos(n)| \geq 0$ since $-1 \leq \cos(n) \leq 1$

larger $\sum \frac{1}{n^3}$ conv ➜ smaller series $\sum \frac{|\cos(n)|}{n^3}$ conv

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+4}{\sqrt{5n^3+8}}$

D.C.T. : $\sum \frac{1}{\sqrt{n}}$ div p−series $p = \frac{1}{2}$ ➜ $\frac{1}{13\, n^{1/2}} = \frac{n}{13\, n^{3/2}} \leq \frac{2n+4}{\sqrt{5n^3+8}}$ for $n \geq 1$

smaller $\frac{1}{13} \sum \frac{1}{n^{1/2}}$ div ➜ larger series $\sum \frac{2n+4}{\sqrt{5n^3+8}}$ div

L.C.T.: $\displaystyle\lim_{n\to\infty} \frac{\frac{2n+4}{\sqrt{5n^3+8}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{(2n+4)\sqrt{n}}{\sqrt{5n^3+8}} = \frac{2}{\sqrt{5}} > 0$

both series $\sum \frac{1}{n^{1/2}}$, $\sum \frac{2n+4}{\sqrt{5n^3+8}}$ diverges

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3+4}$

D.C.T. : $\sum \frac{1}{n^3}$ conv p−series

$\frac{2}{n^3+4} \leq \frac{2}{n^3}$ for $n \geq 1$; larger $2 \sum \frac{1}{n^3}$ conv ➜ smaller series $\sum \frac{2}{n^3+4}$ conv

L.C.T.: $\displaystyle\lim_{n\to\infty} \frac{\frac{2}{n^3+4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2n^3}{n^3+4} = 2 > 0$ ➜ both series $\sum \frac{1}{n^3}$, $\sum \frac{2}{n^3+4}$ converges

(7) $\sum_{n=1}^{\infty} \frac{n}{(n+1)\ 2^n}$

D.C.T. : $\sum \frac{1}{2^n}$ conv geo-series , $r = \frac{1}{2}$

$\frac{n}{(n+1)\ 2^n} \le \frac{n}{(n)\ 2^n} = \frac{1}{2^n}$ for $n \ge 1$; larger $\sum \frac{1}{2^n}$ conv ➜ smaller series $\sum \frac{n}{(n+1)\ 2^n}$ conv

L.C.T.: $\lim_{n\to\infty} \frac{\frac{n}{(n+1)\ 2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$ ➜ both series $\sum \frac{1}{2^n}$, $\sum \frac{n}{(n+1)\ 2^n}$ converges

(8) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4}$

D.C.T. : $\sum \frac{1}{n^4}$ conv p-series ➜ $\frac{\arctan(n)}{n^4} \le \frac{\pi/2}{n^4}$ for $n \ge 1$

larger $\frac{\pi}{2} \sum \frac{1}{n^4}$ conv ➜ smaller series $\sum \frac{\arctan(n)}{n^4}$ conv

L.C.T.: $\lim_{n\to\infty} \frac{\frac{\arctan(n)}{n^4}}{\frac{1}{n^4}} = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} > 0$

both series $\sum \frac{1}{n^4}$, $\sum \frac{\arctan(n)}{n^4}$ converges

(9) $\sum_{n=1}^{\infty} \frac{1 + 2^n}{1 + 3^n}$

D.C.T. : $\sum \left(\frac{2}{3}\right)^n$ conv geo-series , $r = \frac{2}{3}$

$\frac{1+2^n}{1+3^n} \le 2\ \frac{2^n}{3^n}$ for $n \ge 1$; larger $2 \sum \left(\frac{2}{3}\right)^n$ conv ➜ smaller series $\sum \frac{1+2^n}{1+3^n}$ conv

L.C.T.: $\lim_{n\to\infty} \frac{\frac{1+2^n}{1+3^n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{1+2^n}{2^n} \cdot \frac{3^n}{1+3^n} = 1 > 0$ ➜ both series $\sum \left(\frac{2}{3}\right)^n$, $\sum \frac{1+2^n}{1+3^n}$ converges

(10) $\sum_{n=1}^{\infty} \dfrac{n^2+5n}{n^3+n+1}$

D.C.T. : $\sum \dfrac{1}{n}$ div p-series $p = 1 \rightarrow \dfrac{n^2}{3\,n^3} \le \dfrac{n^2+5n}{n^3+n+1}$ for $n \ge 1$

since $n^2 \le n^2+5n$; $3n^3 \ge n^3+n+1$

smaller $\dfrac{1}{3} \sum \dfrac{1}{n}$ div $\rightarrow$ larger series $\sum \dfrac{n^2+5n}{n^3++n+1}$ div

L.C.T.: $\lim\limits_{n\to\infty} \dfrac{\dfrac{n^2+5n}{n^3+n+1}}{\dfrac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{n^3+5n^2}{n^3+n+1} = 1 > 0 \rightarrow$ both series $\sum \dfrac{1}{n}$, $\sum \dfrac{n^2+5n}{n^3+n+1}$ diverges

(11) $\sum_{n=1}^{\infty} \dfrac{5n}{2n^2+5}$

L.C.T. : $\sum \dfrac{1}{n}$ div p-series $p = 1$

L.C.T.: $\lim\limits_{n\to\infty} \dfrac{\dfrac{5n}{2n^2+5}}{\dfrac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{5n^2}{2n^2+5} = \dfrac{5}{2} > 0 \rightarrow$ both series $\sum \dfrac{1}{n}$, $\sum \dfrac{5n}{2n^2+5}$ diverges

(12) $\sum_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$

D.C.T. : $\sum \dfrac{1}{n^{4/3}}$ conv p-series $p = \dfrac{4}{3} \rightarrow \dfrac{n+5}{\sqrt[3]{n^7+n^2}} \le \dfrac{6n}{n^{7/3}}$ for $n \ge 1$

larger series $6 \sum \dfrac{1}{n^{4/3}}$ conv $\rightarrow$ smaller $\sum \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ also conv

(13) $\sum_{n=2}^{\infty} \dfrac{\ln(n)}{n}$

D.C.T. : $\sum \frac{1}{n}$ (div) ➜ $\frac{1}{n} \le \frac{\ln n}{n}$ for $n \ge 3$

$\sum \frac{1}{n}$ (div) ➜ larger $\sum \frac{\ln n}{n}$ also div or

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$$ ➜ both $\sum \frac{1}{n}$ and $\sum \frac{\ln n}{n}$ div

(14) $$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$$

D.C.T. : $\sum \frac{1}{n^2}$ (conv) ➜ $\frac{\arctan n}{n^2+1} \le \frac{\pi/2}{n^2}$ for $n \ge 1$

$\sum \frac{1}{n^2}$ (conv) ➜ smaller $\sum \frac{\arctan n}{n^2+1}$ also conv or

$$\lim_{n\to\infty} \frac{\frac{\arctan n}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} (\arctan n)\left(\frac{n^2}{n^2+1}\right) = \frac{\pi}{2} > 0$$ ➜ both $\sum \frac{1}{n^2}$ and $\sum \frac{\arctan n}{n^2+1}$ conv

(15) $$\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^{3/2}}$$