Subspaces or not ? Justify

> A subset S of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if S is <u>closed</u> under addition and closed under scalar multiplication.

(1) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(2) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-t \\ t \end{pmatrix} \right\}$

(3) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x + y = 0 \right\}$

(4) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \geq 0 \right\}$

(5) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 1 \right\}$

(6) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x = t + 4, y = t, z = t - 2 \right\}$

(7) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| xy = 0 \right\}$

(8) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x - y + z = 0 \right\}$

(9) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x = -2t, y = 5t, z = 3t \right\}$

(10) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 \leq 25 \right\}$

(11) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x + y = 0, z = 0 \right\}$

(12) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| z = 0 \right\}$

(13) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| 2x - 4y + 3z + 5 = 0 \right\}$

(14) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| xy = z \right\}$

(15) $S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = 0 \right\}$

<u>Answers</u>

(1) $t = 0 : \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix}; \vec{v} = b\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S ; \vec{u} + \vec{v} = (a + b)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$;

$k\vec{u} = (ka)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$; both closure hold , therefore S is a subspace of $\mathbb{R}^2$

(2) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$ (no t value to get $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$) ; not a subspace of $\mathbb{R}^2$

closure under addition fails: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S$; but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S$ (no t value to get $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$)

closure under scalar multiplication fails: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S$; but $5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \notin S$ (no t value to get $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$)

Subspaces or not ? Justify

Answers

(3) elements of form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \vec{v} = b\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S ; \vec{u} + \vec{v} = (a+b)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S$;

$k\vec{u} = (ka)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S$; both closure hold , therefore S is a subspace of $\mathbb{R}^2$

(4) not a subspace of $\mathbb{R}^2$: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$; closure under addition holds: $a, c \geq 0 \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \in S$ since $a + c \geq 0$

closure under scalar multiplication fails: $\begin{pmatrix} 5 \\ 6 \end{pmatrix} \in S$; but (-1) $\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \end{pmatrix} \notin S$

(5) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$; Let $\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S$; but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S$ since $1^2 + 1^2 = 2 > 1 ; 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \notin S$ since $5^2 + 0^2 = 25 > 1$;

both closure fail , therefore S is not a subspace of $\mathbb{R}^2$

(6) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$; Let $t = 0 \rightarrow \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}; t = 1 \rightarrow \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \in S$; but $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ -3 \end{pmatrix} \notin S ; 5\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ -10 \end{pmatrix} \notin S$ since $\begin{cases} t + 4 = 20 \Rightarrow t = 16 \\ t = 0 \end{cases}$;

since $\begin{cases} t + 4 = 9 \Rightarrow t = 5 \\ t = 1 \end{cases}$ contradictory t's

both closure fail , therefore S is not a subspace of $\mathbb{R}^2$

(7) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$; Let $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \in S$; but $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \notin S$ since $(1)(1) \neq 0 ; 5\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix} \in S$;

closure under addition fails , therefore S is not a subspace of $\mathbb{R}^2$

Note: scalar multiplication works; $(x, y, z) \in S \Rightarrow xy = 0, k(x, y, z) \in S$ since

$(kx)(ky) = k^2(xy) = k^2(0) = 0.$

(8) parametric form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s - t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \vec{v} = c\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in S$;

$\vec{u} + \vec{v} = (a+c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (b+d)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in S ; k\vec{u} = (ka)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (kb)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in S$; both closure hold , then S is a subspace of $\mathbb{R}^3$

(9) $t = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}; \vec{v} = b\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \in S ; \vec{u} + \vec{v} = (a+b)\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \in S ; k\vec{u} = (ka)\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \in S$;

both closure hold , therefore S is a subspace of $\mathbb{R}^3$

(10) $(0, 0, 0) \in S$; Let $\vec{u} = (0, 0, 5); \vec{v} = (0, 5, 0) \in S ; \vec{u} + \vec{v} = (0, 5, 5) \notin S$ since $5^2 + 5^2 = 50 > 25$;

$4\vec{u} = (0, 0, 20) \notin S$ since $20^2 = 400 > 25$; both closure fail , therefore S is not a subspace of $\mathbb{R}^3$

Subspaces or not ? Justify

Answers

(11) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in S$; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \vec{v} = b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in S ; \vec{u} + \vec{v} = (a+b)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in S ; k\vec{u} = (ka)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in S$;

both closure hold , therefore S is a subspace of $\mathbb{R}^3$

(12) parametric form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$; Let $\vec{u} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \vec{v} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in S$;

$\vec{u} + \vec{v} = (a+c)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b+d)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in S ; k\vec{u} = (ka)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (kb)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in S$; both closure hold , then S is a subspace of $\mathbb{R}^3$

(13) $\left(0,0,0\right) \notin S$; Let $\vec{u} = \left(0,0,-\frac{5}{3}\right); \vec{v} = \left(-\frac{5}{2},0,0\right) \in S ; \vec{u} + \vec{v} = \left(-\frac{5}{2},0,-\frac{5}{3}\right) \notin S$ since $2\left(-\frac{5}{2}\right) - 4(0) + 3\left(-\frac{5}{3}\right) + 5 \neq 0$;

$3\vec{u} = \left(0,0,-5\right) \notin S$ since $2(0) - 4(0) + 3(-5) + 5 \neq 0$;

both closure fail , therefore S is not a subspace of $\mathbb{R}^3$

(14) $\left(0,0,0\right) \in S$; Let $\left(1,5,5\right); \left(3,4,12\right) \in S ; \left(1,5,5\right) + \left(3,4,12\right) = \left(4,9,17\right) \notin S$ since $(4)(9) \neq 17$;

$3\left(1,5,5\right) = \left(3,15,15\right) \notin S$ since $(3)(15) \neq 15$;

both closure fail , therefore S is not a subspace of $\mathbb{R}^3$

(15) this is just the $\vec{O} = \left(0,0,0\right) \in S$; $\left(0,0,0\right) + \left(0,0,0\right) = \left(0,0,0\right) \in S ; k\vec{O} = \vec{O} \in S$;

both closure hold , therefore S is a subspace of $\mathbb{R}^3$