

Review Problems # 5

(1) Determine whether the following sets are LI or LD. Justify your answers.

(a) $S_1 = \{(2, 1, -4)\}$; (b) $S_2 = \{(2, 1, -4), (0, 0, 0)\}$

(c) $S_3 = \{(3, 1, 2), (6, 2, 4)\}$; (d) $S_4 = \{(1, 0, 1, 1), (2, 1, 0, 3), (1, -1, 2, 5)\}$

(2) Write (if possible) \vec{u} as a L.C. of \vec{v} and \vec{w} where

$$\vec{u} = (11, 3, 22, 11) , \vec{v} = (-1, 2, 3, 4) \text{ and } \vec{w} = (3, -1, 2, -1)$$

(3) Determine if $\vec{v} = (1, -3, 2)$ can be written as a L.C. of

$$\vec{u}_1 = (1, 2, -1) , \vec{u}_2 = (3, 5, 2) \text{ and } \vec{u}_3 = (4, 7, 1)$$

(4) Find a basis for each of the subspaces

(a) $\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 3z = 0\}$; (b) $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + 5y = 0, z = 0, w = 0\}$

(5) Does $(8, -3, -6)$ belong to $\text{span}\{(1, 0, 3), (-2, 1, 4)\}$? Why or why not ?

(6) Determine the span of each of the following sets of vectors

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} \right\}$; (b) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 9 \end{pmatrix} \right\}$; (c) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

(7)

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 2 & 0 & 1 \\ 2 & 4 & 0 & 3 & 1 \\ 5 & 10 & 6 & 3 & 4 \\ 2 & 4 & 4 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

find : (a) the Null Space of A , a basis for it and the dimension

(b) a basis for Column Space of A and the dimension

(c) write the non-basic column vectors as L.C.'s of Basic Column vectors

(d) a basis for Row Space of A and the dimension

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(8) Let $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ represent the column vectors of

$$A = \begin{pmatrix} 1 & -2 & 4 & -5 \\ 2 & 3 & 7 & 4 \\ 0 & 3 & 4 & 6 \\ 1 & 4 & 6 & 7 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ the REEF of } A$$

- (a) Express \vec{u}_4 as a L.C. of $\vec{u}_1, \vec{u}_2, \vec{u}_3$
 (b) which one of the 4 vectors is not a L.C. of the other three?
 (c) Are the column vectors of A LI or LD? Justify
 (d) Do the column vectors of A span \mathbb{R}^4 ? Why or why not?
 (e) Find a basis for the Column Space of A
 (f) Find a basis for the Null Space of A
 (g) Find a basis for Row Space of A

(9) Find the solution space of $\begin{cases} x_1 - x_2 + x_3 - 2x_4 = 0 \\ x_1 + x_2 - 2x_3 + x_4 = 0 \end{cases}$

(10) Show that $\text{span} \left\{ (1, 0, -3), (2, -1, 4) \right\} = \text{span} \left\{ (5, -2, 5), (3, -1, 1) \right\}$

(11) (a) Write $\begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}$ as a L.C. of $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

(b) Does $S_1 = \left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \right\}$ span \mathbb{R}^3 ? Explain

(c) Find a set S_2 of 3 vectors in \mathbb{R}^3 which does span \mathbb{R}^3 . (Justify your choice)

(12) Find what values of c is the set of vectors $\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix} \right\}$ LD?

(13) Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}; \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}; \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is LD
 (b) Explain why these vectors do not span \mathbb{R}^4

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(14) Prove that the following subsets are not subspaces of the given vector space

$$(a) S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 5x = 2yz \right\}; \quad (b) S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x = t + 1, y = t - 3, z = t, w = t \right\}$$

$$(c) S_3 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid xyzw = 0 \right\}; \quad (d) S_4 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid 3xy = 4zw \right\}; \quad (e) S_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0 \right\}$$

(15) Prove that the following are subspaces of the given vector space

$$(a) S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + 3y - z = 0 \right\}; \quad (b) S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x - 2z = 0 \right\}; \quad (c) S_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x = 3y \right\}$$

(16) Find a basis for subspaces in # 15 . State the dimension of each.

(17) (a) Prove $S = \text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is a subspace of \mathbb{R}^4 where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in \mathbb{R}^4

(b) Is the dimension of $S = 3$? (Justify)

Answers

$$(1 a) \text{ LI}; c_1(2, 1, -4) = (0, 0, 0) \Rightarrow c_1 = 0; \quad (1 b) \text{ LD}; 0(2, 1, -4) + \underset{\text{non-zero}}{5}(0, 0, 0) = (0, 0, 0)$$

$$(1 c) \text{ LD}; 2(3, 1, 2) - (6, 2, 4) = (0, 0, 0); \quad (1 d) \text{ LI}; \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \sim \begin{array}{c} \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \\ c_1 = c_2 = c_3 = 0 \text{ uniquely} \end{array} \end{array}$$

$$(2) \vec{u} = 4\vec{v} + 5\vec{w}$$

$$(3) \vec{v} \text{ is not a L.C. of } \vec{u} \text{ 's} \rightarrow \begin{array}{c} \begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ 1 & 3 & 4 & 1 \\ 2 & 5 & 7 & -3 \\ -1 & 2 & 1 & 2 \end{array} \rightarrow \begin{array}{ccc|c} & & & \\ 0 & 0 & 0 & \neq 0 \end{array} \end{array}$$

$$(4 a) \left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is one possibility}; \quad (4 b) \left\{ (-5, 1, 0, 0) \right\} \text{ is one possibility}$$

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Answers

(5) yes! $2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -6 \end{pmatrix} \Rightarrow (8, -3, -6)$ is a L.C. of the 2 vectors

(6 a) plane in \mathbb{R}^3 with equation $7x - 16y - 5z = 0$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ still a plane in \mathbb{R}^3

(6 b) line in \mathbb{R}^3 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$; (6 c) LI vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = r \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ a 3-d subspace of \mathbb{R}^4

(7 a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2s - \frac{3}{2}t - \frac{1}{2}r \\ s \\ \frac{3}{4}t - \frac{1}{4}r \\ t \\ r \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; (7 b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 6 \\ 4 \end{pmatrix} \right\}$; $d = 2$
 basis : $\left\{ (-2, 1, 0, 0, 0), (-\frac{3}{2}, 0, \frac{3}{4}, 1, 0), (-\frac{1}{2}, 0, -\frac{1}{4}, 0, 1) \right\}$; $d=3$

(7 c) $\begin{pmatrix} 2 \\ 4 \\ 10 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix} ; \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 2 \\ 0 \\ 6 \\ 4 \end{pmatrix} ; \begin{pmatrix} 1 \\ 1 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 5 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2 \\ 0 \\ 6 \\ 4 \end{pmatrix}$; (7 d) $\left\{ (1, 2, 2, 0, 1), (2, 4, 0, 3, 1) \right\}$ or $\left\{ (1, 2, 0, \frac{3}{2}, \frac{1}{2}), (0, 0, 1, -\frac{3}{4}, \frac{1}{4}) \right\}$

(8 a) $\vec{u}_4 = -\vec{u}_1 + 2\vec{u}_2 + 0\vec{u}_3$; (8 b) \vec{u}_3 (it cannot be isolated since it has a "0" coefficient)

(8 c) LD (solution for c 's is parametric ; $c_4 = t$)

(8 d) No! Span { col vectors of A } is a 3-d subspace of \mathbb{R}^4 , not the entire space

(8 e) $\{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \}$;

(8 f) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$; basis = $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$; (8 g) $\left\{ (1, -2, 4, -5), (2, 3, 7, 4), (0, 3, 4, 6) \right\}$
 or first 3 row vectors of R

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Answers

$$(9) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}; \text{ a 2-d subspace of } \mathbb{R}^4 \text{ (same as Null Space } A \text{)}$$

(10) both are planes whose equation is $3x + 10y + z = 0$

$$(11 \text{ a}) 3 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}; \quad (11 \text{ b}) \text{ no! } \text{span} \left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\},$$

\therefore a plane in \mathbb{R}^3 , not the entire space

(11 c) any 3 LI vectors will span \mathbb{R}^3 ; for example $\{(1,0,0), (0,1,0), (0,0,1)\}$; check $\det A \neq 0$

$$(12) \det \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & c \end{pmatrix} = (-1)(c-2) - 1(1) = -c + 1 = 0 \text{ when } c = 1; \text{ if } \det A = 0, A \sim R$$

where R has a row of zeros \Rightarrow parametric solution for c 's \Rightarrow vectors are LD

(13 a) $\det (\vec{v}_1 \vec{v}_2 \vec{v}_3 \vec{v}_4) = 0 \Rightarrow$ vectors are LD i.e. related by a dependency equation

(13 b) you need 4 LI vectors to span the entire \mathbb{R}^4

$$(14 \text{ a}) \text{ addition fails: } \vec{u} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \in S_1; \vec{v} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} \in S_1 \text{ but } \vec{u} + \vec{v} = \begin{pmatrix} 8 \\ 8 \\ 6 \end{pmatrix} \notin S_1$$

$5(2)=2(5)(1) \qquad 5(6)=2(3)(5) \qquad 5(8) \neq 2(8)(6)$

(14 b) $(0,0,0,0) \notin S_2$ since $x = t + 1 = 0 \Rightarrow t = -1, y = t - 3 = 0 \Rightarrow t = 3, z = t = 0, w = t = 0$
contradictory t 's; also you could show addition or scalar multiplication fails

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(14 c) addition fails: $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} \in S_3$; $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \in S_3$ but $\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix} \notin S_3$

$(1)(2)(0)(3)=0$ $(0)(1)(2)(3)=0$ $(1)(3)(2)(6)\neq 0$

(14 d) addition fails: $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \in S_4$; $\vec{v} = \begin{pmatrix} 4 \\ 1 \\ 1 \\ 3 \end{pmatrix} \in S_4$ but $\vec{u} + \vec{v} = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix} \notin S_4$

$3(1)(0)=(4)(2)(0)$ $3(4)(1)=(4)(1)(3)$ $3(5)(1)\neq(4)(3)(3)$

(14 e) addition works: $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S_5$; $\vec{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \in S_5$ and $\vec{u} + \vec{v} = \begin{pmatrix} x+p \\ y+q \\ z+r \end{pmatrix} \in S_5$

$x \geq 0, y \geq 0, z \geq 0$ $p \geq 0, q \geq 0, r \geq 0$ $x+p \geq 0, y+q \geq 0, z+r \geq 0$

scalar multiplication fails: $(1, 2, 3) \in S_5$ but $-2(1, 2, 3) = (-2, -4, -6) \notin S_5$

(15 a) Let: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s+t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = s \vec{a}_1 + t \vec{a}_2$; where basis vectors are $\vec{a}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Let: $\vec{u} = a \vec{a}_1 + b \vec{a}_2 \in S_1$ and $\vec{v} = c \vec{a}_1 + d \vec{a}_2 \in S_1$

$\left\{ \begin{array}{l} \text{addition : } \vec{u} + \vec{v} = (a+c) \vec{a}_1 + (b+d) \vec{a}_2 \in S_1 \\ \text{scalar multiplication : } k\vec{u} = (ka) \vec{a}_1 + (kb) \vec{a}_2 \in S_1 \end{array} \right\} \rightarrow S_1 \text{ is a subspace of } \mathbb{R}^3$

(15 b) Let: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ s \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = t \vec{a}_1 + s \vec{a}_2$; where basis vectors are $\vec{a}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

proof: same as in (15 a)

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(15 c) Let: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t \\ -t \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \end{pmatrix} = t \vec{a}_1$; where basis vector is $\vec{a}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Let: $\vec{u} = a \vec{a}_1 \in S_3$ and $\vec{v} = b \vec{a}_1 \in S_3$

$$\left\{ \begin{array}{l} \text{addition : } \vec{u} + \vec{v} = (a + b) \vec{a}_1 \in S_3 \\ \text{scalar multiplication : } k\vec{u} = (ka) \vec{a}_1 \in S_3 \end{array} \right\} \rightarrow S_3 \text{ is a subspace of } \mathbb{R}^3$$

(16 a) $\{(-3, 1, 0), (1, 0, 1)\}$, $d = 2$; (16 b) $\{(2, 0, 1), (0, 1, 0)\}$, $d = 2$; (16 c) $\{(3, 1)\}$, $d = 1$

(17 a) Let: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = r \vec{v}_1 + s \vec{v}_2 + t \vec{v}_3 \in S$

$$\left\{ \begin{array}{l} \vec{u} = a \vec{v}_1 + b \vec{v}_2 + c \vec{v}_3 \in S \\ \vec{v} = d \vec{v}_1 + e \vec{v}_2 + f \vec{v}_3 \in S \\ \vec{u} + \vec{v} = (a + d) \vec{v}_1 + (b + e) \vec{v}_2 + (c + f) \vec{v}_3 \in S \\ \text{and } k\vec{u} = (ka) \vec{v}_1 + (kb) \vec{v}_2 + (kc) \vec{v}_3 \in S \end{array} \right. \rightarrow S \text{ is a subspace of } \mathbb{R}^4$$

(17 b) No! dimension of $S = 3$ only if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI and we don't know that.

Any of these options is possible:

- The dimension is either 0 (if all vectors are $\vec{0}$,
- 1 (if all vectors are multiples) ,
- 2 (if $\vec{v}_3 =$ a L.C. of \vec{v}_1 and \vec{v}_2) ,
- or 3 (if vectors are LI)