Review Problems # 3

(1) Given $E=\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ d & e & f \end{pmatrix}$; $F=\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \\ d & e & f \end{pmatrix}$; $K=\begin{pmatrix} 2a & 2b & 2c \\ 3 & 6 & 9 \\ d & e & f \end{pmatrix}$, if det E = 31 ,

find (a) det F ; (b) det K ; (c) det $(2E)$; (d) det $\left(E^{-1}\right)$; (e) det (EF) ; (f) det $(K-E)$

(2) Let $A=\begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$ and det $A=2$;

(i) find: (a) det $\left(AA^t\right)$; (b) det $\left(-A^2\right)$; (c) det $(3A)^{-1}$; (d) det $\begin{pmatrix} a & b & c \\ 3x & 3y & 3z \\ r+2x & s+2y & t+2z \end{pmatrix}$; (e) C_{32}

(ii) what is the solution to $A\vec{x}=\vec{O}$?

(iii) What is the rank of A ?

(3) Let $A=\begin{pmatrix} 3 & 0 & -6 \\ -4 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B=\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}$

(a) find det A , adj A and A^{-1} , $(A)(\text{adj } A)$, det $(\text{adj } A)$

(b) find A^{-1} using the $[A \mid \text{I}]$ process

(c) use A^{-1} to $AX=B$ where $B=\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}$

(d) find x , y , z using Cramer's Rule

(4) Evaluate det $\begin{pmatrix} 3 & 1 & -2 & 4 \\ 1 & -2 & -3 & 2 \\ -2 & 4 & 8 & 3 \\ 0 & 2 & -2 & 4 \end{pmatrix}$

(5) Given $C=\begin{pmatrix} 2 & 3 & 2 & 1 \\ 4 & 6 & 0 & -2 \\ 3 & -1 & 0 & 5 \\ -2 & 2 & -4 & 3 \end{pmatrix}$, find (a) C_{43} , (b) det C

(6) Find the following by inspection.

(a) det $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 5 & 7 & -3 & 0 \\ 2 & 9 & 8 & 5 \end{pmatrix}$; (b) det $\begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) det $\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 7 & 8 \\ -1 & -2 & 2 & 3 \\ 4 & 8 & 9 & 8 \end{pmatrix}$

<u>**Review Problems # 3**</u>

(7) Let $A=\begin{pmatrix}1 & 3 & 10\text{k}\\ 2 & 1 & -5\\ 1 & -2 & 7\end{pmatrix}$, (a) find K such that det A = 0
(b) find K such that $AX=0$ has nontrivial solutions
(c) find K such that $AX=B$ has a unique solution

(8) Let $A^{-1}=\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$; $B^{-1}=\begin{pmatrix}2 & -1\\ 3 & 0\end{pmatrix}$, find (a) A , (b) $(AB)^{-1}$, (c) $(AB)^t$, (d) A^2 ; (e) B^{-2}

(9) Given $A=\begin{pmatrix}2 & 4\\ -1 & -1\end{pmatrix}$, find $A^{-2}-3A+4\text{I}$; (10) Find A if $(\text{I}+A)^{-1}=3\begin{pmatrix}2 & 2\\ 3 & 2\end{pmatrix}$

(11) Simplify: (a) $\left(3C^{-1}AB\right)^{-1}\left(C^{-1}A^2\right)$, (b) det $\left(3C^{-1}A^tB\right)$; (c) $\left(3C^{-1}A^tB\right)^t$
Assume A , B , C are 3x3 invertible matrices

(12) (a) $ABX=A+B$ for X assuming A^{-1} , B^{-1} $\exists$; (b) $AX+B=X$, assuming $(\text{I}-A)^{-1}$ $\exists$

(13) Let $A=\begin{pmatrix}0 & -5\\ 2 & 4\end{pmatrix}$; $B=\begin{pmatrix}3 & -4\\ 2 & -1\end{pmatrix}$, show (a) $(A+B)^{-1}\neq A^{-1}+B^{-1}$, (b) $A^2-B^2\neq(A-B)(A+B)$

(14) If $A^2=A^t$, find all possible values for det A ; (15) Prove $(\text{K}A)^{-1}=\dfrac{1}{\text{k}}A^{-1}$

(16) Find A , B such that A^{-1} , B^{-1} $\exists$ but $(A+B)^{-1}$ $\nexists$
(17) Find A , B such that det $(A+B)\neq$ det $A+$ det B

<u>Answers</u>:
(1 a) -31 ; (b) 186 ; (c) 248 ; (d) $\frac{1}{31}$; (e) (det E)(det F)= -961 ; (f) 0

(2 i) (a) 4 ; (b) $(-1)^3(\text{det } A)^2=-4$; (c) $\frac{1}{54}$; (d) -6 ; (e) $-\text{M}_{32}=-(at-cr)$

(2 ii) $X=0$, trivial solution only since A^{-1} exists ;

(2 iii) rank $A=3$ since A is reducible to I and I has 3 leading ones

(3 a) det $A=9$; adj $A=\begin{pmatrix}-3 & -6 & -6\\ 9 & 12 & 21\\ -3 & -3 & -3\end{pmatrix}$; $A^{-1}=\dfrac{1}{9}\begin{pmatrix}-3 & -6 & -6\\ 9 & 12 & 21\\ -3 & -3 & -3\end{pmatrix}$; $A(\text{adj } A)=\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}$;

det (adj A) $=9^2=81$; (3 b) $A^{-1}=\dfrac{1}{9}\begin{pmatrix}-3 & -6 & -6\\ 9 & 12 & 21\\ -3 & -3 & -3\end{pmatrix}$ as in parts (a)

(3 c) $X=A^{-1}B=\begin{pmatrix}-8\\ 30\\ -5\end{pmatrix}$; (d) same as (c) ; (4) -260 ; (5 a) 44 ; (5 b) -524

Review Problems # 3

Answers:

(6) (a) -120 ($\triangle$ matrix) ;

(b) $(-1)(-2)=2$ (row interchange $\rightarrow$ diagonal matrix) ; (c) 0 (c_1 and c_2 are multiples)

(7) (a) $k=-\frac{6}{5}$; (b) when det $A=0$, i.e. $k=-\frac{6}{5}$; (c) det $A\neq 0$, $k\neq -\frac{6}{5}$

(8) (a) $A=\frac{1}{11}\begin{pmatrix}1 & 2\\ -4 & 3\end{pmatrix}$; (b) $\begin{pmatrix}2 & -5\\ 9 & -6\end{pmatrix}$; (c) $\frac{1}{33}\begin{pmatrix}-6 & -9\\ 5 & 2\end{pmatrix}$; (d) $\frac{1}{121}\begin{pmatrix}-7 & 8\\ -16 & 1\end{pmatrix}$; (e) $\begin{pmatrix}1 & -2\\ 6 & -3\end{pmatrix}$

(9) $\begin{pmatrix}-\frac{11}{4} & -13\\ \frac{13}{4} & 7\end{pmatrix}$; (10) $\begin{pmatrix}-\frac{4}{3} & \frac{1}{3}\\ \frac{1}{2} & -\frac{4}{3}\end{pmatrix}$; (11 a) $\frac{1}{3}B^{-1}A^{-1}\left(CC^{-1}\right)A^2=\frac{1}{3}B^{-1}A^{-1}\left(\mathrm{I}A^2\right)=\frac{1}{3}B^{-1}A$;

(11 b) $\dfrac{27\ (\det A)\ (\det B)}{\det C}$; (11 c) $3B^tA\left(C^{-1}\right)^t$

(12 a) $X=B^{-1}+B^{-1}A^{-1}B$ or $B^{-1}\left(\mathrm{I}+A^{-1}B\right)$ or $B^{-1}+(AB)^{-1}B$

(12 b) $X=(\mathrm{I}-A)^{-1}B$ or $-(A-\mathrm{I})^{-1}B$; (13 a) $(A+B)^{-1}=\frac{1}{45}\begin{pmatrix}3 & 9\\ -4 & 3\end{pmatrix}\neq A^{-1}+B^{-1}=\frac{1}{10}\begin{pmatrix}2 & 13\\ -6 & 6\end{pmatrix}$

(13 b) $A^2-B^2=\begin{pmatrix}-11 & -12\\ 4 & 13\end{pmatrix}\neq (A-B)(A+B)=\begin{pmatrix}-13 & 24\\ 20 & 15\end{pmatrix}$

(14) $\left(\det A^2\right)=\det A^t$ (take the "det" of both sides) ; $(\det A)^2=\det A$ (similar $x^2=x$)

$(\det A)^2-\det A=0$ (factor) ; $\det A\ (\det A-1)=0\Rightarrow \det A=0$ or 1

(15) $(kA)\left(\frac{1}{k}A^{-1}\right)=\left(k.\frac{1}{k}\right)AA^{-1}=\mathrm{I}$; also $\left(\frac{1}{k}A^{-1}\right)(kA)=\mathrm{I}$ so that $(kA)^{-1}=\frac{1}{k}A^{-1}$

" Study " your Inverse theorems $\rightarrow$ this is only one of them!

(16) $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$; $B=\begin{pmatrix}-1 & -2\\ -3 & -4\end{pmatrix}$; $A^{-1}=\frac{1}{-2}\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$; $B^{-1}=\frac{1}{-2}\begin{pmatrix}-4 & 2\\ 3 & -1\end{pmatrix}$;

but $A+B=0$; $0^{-1}\ \nexists$

(17) almost any A , B will do! , for example :

$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$; $B=\begin{pmatrix}-2 & 4\\ 3 & 1\end{pmatrix}$, det $A=-2$, det $B=-14$

det $(A+B)=$ det $\begin{pmatrix}-1 & 6\\ 6 & 5\end{pmatrix}=-41\neq$ det $A+$ det B