Review Problems # 1

(1) Given $\begin{cases} 2x_1 + 4x_2 + 2x_3 - x_4 + 7x_5 = 9 \\ x_1 + 2x_2 - 4x_3 + 3x_4 + 2x_5 = 5 \\ 3x_1 + 6x_2 - 2x_3 + 2x_4 + 9x_5 = 14 \end{cases}$

(a) find a general solution for the system using R.R.E.F. Indicate which row operations are used.

(b) find a particular solution where $\left(x_1, x_2, x_3, x_4, x_5\right) = \left(a, 10, b, 20, 30\right)$. (i.e. a = ? , b = ?)

(2) What kind of solutions do the following systems have ? Why ?

(a) $\begin{cases} 3x_1 - x_2 + 5x_3 - x_4 = 0 \\ 9x_1 - 5x_2 + x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 + x_4 = 0 \end{cases}$ (b) $\begin{cases} 5x_1 - 2x_2 + x_3 = 0 \\ -4x_2 + 5x_3 = 0 \\ 9x_3 = 0 \end{cases}$

(3) Find the general solution and 2 particular solutions for the following system by reducing the matrix to an R.R.E.F.

$$\begin{cases} 4x_1 - 4x_2 + 0x_3 + 4x_4 - 8x_5 = -4 \\ 2x_1 - 2x_2 + 0x_3 + x_4 - 2x_5 = -3 \\ -6x_1 + 6x_2 + 0x_3 - 2x_4 + 4x_5 = 10 \end{cases}$$

(4) solve the following system using <u>Back Substitution</u>

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

(5) (a) Find a consistency condition for the system:

$$\begin{cases} 3x + 4y - z = a \\ x + 2y + 5z = b \\ 7x + 10y + 3z = c \end{cases}$$

(b) Is the system consistent for a = b = c = 1 ?

(c) If not, change the value of c so that the system becomes consistent.

(d) Will the solution to the system ever be unique ? why or why not ?

(6) Show that the system

$\begin{cases} x - z = a \\ 2x + y + 3z = b \\ 3x + y = c \end{cases}$ is consistent for all values of a , b , c .

(7) Find all value(s) of k such that the solution to the following system is (a) unique

(b) parametric (c) non existent $\begin{cases} kx + y + z = 0 \\ x + y + 4z = 0 \\ x + y + k^2 z = 0 \end{cases}$

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(8) Repeat the instructions in # 7 for the system: $\begin{cases} x + 3y - 2z = 0 \\ x + (\mathrm{k}+2)y - z = -2 \\ 2x + 6y + \mathrm{k}z = 2 \end{cases}$

(9) Find A such that $\left(2A^t - B\right)^t = C + A$ where $B = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 0 & 4 \\ 2 & 3 \end{pmatrix}$

(10) Find s and t such that $A = -A^t$ where $A_{2\times 2} = \begin{pmatrix} s & t^2 \\ 3t-4 & s \end{pmatrix}$

(11) Find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} = \mathrm{I}$

(12) Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

Verify that (a) $A(BC) = (AB)C$; (b) $(AC)^{-1} = C^{-1}A^{-1}$; (c) $\left(B^t\right)^{-1} = \left(B^{-1}\right)^t$

(13) Find $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $UA = 0$ given that (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

(14) Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ -1 & 5 \end{pmatrix}$

Verify that (a) $(A+B)^2 \neq A^2 + 2AB + B^2$; (b) $(A+B)^{-1} \neq A^{-1} + B^{-1}$

(15) A company has 3 refineries R_1 , R_2 , R_3

R_1 produces 20 gallons of heating oil, and 5 gallons of gasoline per barrel of petroleum.
R_2 produces 4 gallons of heating oil, and 6 gallons of gasoline per barrel of petroleum.
R_3 produces 4 gallons of heating oil and 11 gallons of gasoline per barrel of petroleum.

How many barrels of petroleum should each refinery produce to meet a demand for 500 gallons of heating oil, and 1125 gallons of gasoline ?

(a) Define your variables (in words)
(b) Set up a matrix describing the system of equations needed to solve the problem
(c) Assuming that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{5}(t-75) \\ 200-2t \\ t \end{pmatrix}$ is a general solution of the system, find an interval for t for appropriate particular solutions.
(d) Find <u>one</u> appropriate particular solution.

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Answers:

(1 a)
$$\left[\begin{array}{ccccc|c|c} \boxed{1} & 2 & -4 & 3 & 2 & 5 & 9 \\ 2 & 4 & 2 & -1 & 7 & 9 & 23 \\ 3 & 6 & -2 & 2 & 9 & 14 & 32 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 \; r & x_3 & x_4 \; s & x_5 \; t & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & \frac{1}{5} & \frac{16}{5} & \frac{23}{5} \\ 0 & 0 & \boxed{1} & -\frac{7}{10} & \frac{3}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$
RREF

(1 a) $\Rightarrow$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{23}{5} - 2r - \frac{1}{5}s - \frac{16}{5}t \\ r \\ \frac{7}{10}s - \frac{3}{10}t - \frac{1}{10} \\ s \\ t \end{pmatrix}$$
; (1 b) $x_1 = a = -\frac{577}{5}$; $x_3 = b = \frac{49}{10}$

(2 a) parametric ! maximum # leading ones (or pivots) = 3 < # unknowns = 4 ;
there is at least <u>one</u> parameter

(2 b) <u>Unique</u> ! $x_3 = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$ # leading ones (pivots) = # unknowns

(3)
$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 \; r & x_3 \; s & x_4 & x_5 \; t & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} r-2 \\ r \\ s \\ 2t+1 \\ t \end{pmatrix}$$
<u>2 particular solutions</u> – many possible answers:

for example: $r = 0 = s = t \Rightarrow \left(-2,0,0,1,0\right)$ or $r = 0, s = 0, t = 1 \Rightarrow \left(-2,0,0,3,1\right)$ and <u>so on</u> !

(4) $z = -5 \Rightarrow y - 5z = 4 \Rightarrow y = 5(-5) + 4 = -21 \Rightarrow x = 2y - 7z + 3 = 2(-2) - 7(-5) + 3 = -4$

(5 a) 2a+b-c = 0 ; (b) No! 2+1-1 = 2 ≠ 0 ; (c) c = 3
(d) never ! # leading ones (pivot) = 2 < # unknowns = 3

(6)
$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & a \\ 0 & \boxed{1} & 5 & -2a+b \\ 0 & 0 & \boxed{-2} & -a-b+c \end{array}\right] \end{array}$$
$\Rightarrow$ has a unique solution for all a , b , c

(3 non-zero pivots (leading ones) = 3 # unknowns)

(7) (a) k $\neq$ 1 , k $\neq$ ± 2 ; (b) k = 1 , ± 2 ; (c) Never ! (Homogeneous System)

(8) (a) k $\neq$ 1 , -4 ; (b) none ; (c) k = -4 , 1

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Answers:

(9)

$$A=\begin{pmatrix}4 & -3\\ -2 & 5\\ 3 & 3\end{pmatrix}$$

(10) s = 0 , t = -4 or s = 0 , t = 1

$$A=\begin{pmatrix}0 & 16\\ -16 & 0\end{pmatrix} \qquad A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

(11) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\frac{3}{10} & \frac{1}{10}\\ -\frac{4}{10} & \frac{2}{10}\end{pmatrix}$

easy way to solve: Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}; B=\begin{pmatrix}2 & -1\\ 4 & 3\end{pmatrix} \Rightarrow AB=\mathrm{I} \Rightarrow ABB^{-1}=\mathrm{I}B^{-1} \Rightarrow A\mathrm{I}=B^{-1} \Rightarrow A=B^{-1}$

(12 a) LHS = A(BC) = ? and RHS = (AB)C = ? they should be equal ; (b) and (c) similar to (a)

(13 a) $U=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$; solve : $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \Rightarrow \begin{cases}\left.\begin{matrix}a+3b=0\\ 2a+4b=0\end{matrix}\right\} \Rightarrow a=0, b=0\\ \left.\begin{matrix}c+3d=0\\ 2c+4d=0\end{matrix}\right\} \Rightarrow c=0, d=0\end{cases}$ or Easy Way for (a) :

$$UA=0 \Rightarrow UAA^{-1}=0A^{-1} \Rightarrow U\mathrm{I}=0 \Rightarrow U=0$$

(13 b) Here you have to do the equations since $A^{-1} \nexists, U=\begin{pmatrix}-2t & t\\ -2s & s\end{pmatrix}$

(14 a) $(A+B)^2=\begin{pmatrix}14 & -11\\ 22 & 47\end{pmatrix} \neq A^2+2AB+B^2=\begin{pmatrix}15 & -14\\ 13 & 46\end{pmatrix}$

(14 b) $(A+B)^{-1}=\frac{1}{30}\begin{pmatrix}7 & 1\\ -2 & 4\end{pmatrix} \neq A^{-1}+B^{-1}=\frac{1}{7}\begin{pmatrix}2 & 1\\ -3 & 2\end{pmatrix}+\frac{1}{10}\begin{pmatrix}5 & 0\\ 1 & 2\end{pmatrix}=\begin{pmatrix}\frac{11}{14} & \frac{1}{7}\\ -\frac{23}{70} & \frac{17}{35}\end{pmatrix}$

(15 a) x = # barrels of petroleum produced by R_1 , etc..

(15 b)

(15 b) $\begin{array}{ccc|c} x & y & z & \\ 20 & 4 & 4 & 500\\ 5 & 6 & 11 & 1125\end{array} \Rightarrow$ general solution : $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\frac{1}{5}(t-75)\\ 200-2t\\ t\end{pmatrix}$

(15 c) $75 \le t \le 100$; (15 d) t = 75 ➡ (0, 50, 75) etc.. (positive integers only)