Solutions to Selected Questions from Denis Sevee's *Vector Geometry.*

(Updated 2014-11-27)

Denis Sevee's Vector Geometry notes appear as "Chapter 5" in the current custom textbook used at John Abbott College for the course 201-NYC-05 (Linear Algebra for Science Students). Answers to most of the questions are at the end of chapter 5, but occasional answers are omitted and no full solutions are available. Therefore I have written up some additional answers and some solutions here. Please let me know if you find a mistake, or if you have an additional solution to include. – A. McLeod

Chapter 5.1

1. $\overrightarrow{AB} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

   so $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

   so finally, $\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = B$

7. $||\mathbf{u} - \mathbf{v}|| = \sqrt{(1-k)^2 + (2-(k+1))^2 + (4-(k+2))^2} = \sqrt{6 - 8k + 3k^2}$

   This expression is minimized when $6 - 8k + 3k^2$ is minimized. Using techniques from Calculus 1, locate the minimum value by taking the derivative and setting it equal to zero: $-8 + 6k = 0$ so $k = 4/3$. (Note: we should also confirm that it's a minimum rather than a maximum by checking that the second derivative is > 0.)

11. First consider the right triangle with vertices $(0,0,0)$, $(x_1,0,0)$ and $(x_1,x_2,0)$. We can ignore the third dimension (since the third coordinate is 0 in each case), so this is a triangle with legs of length x_1 and x_2, and so by the Pythagorean Theorem the length of the hypotenuse is $\sqrt{x_1^2 + x_2^2}$.

    Then consider the triangle with vertices $(0,0,0)$, $(x_1,x_2,0)$ and (x_1,x_2,x_3). Its base is the hypotenuse of the earlier triangle, its height is x_3, and its hypotenuse is $\mathbf{v}$. So, using Pythagoras again:

    $||\mathbf{v}||^2 = (\sqrt{x_1^2 + x_2^2})^2 + x_3^2$

    $||\mathbf{v}||^2 = x_1^2 + x_2^2 + x_3^2$

    $||\mathbf{v}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

12. f. True. Recall that in general, $||\mathbf{x}||^2 = \mathbf{x}^T\mathbf{x}$ and $(AB)^T = B^TA^T$. So $||A\mathbf{u}||^2 = (A\mathbf{u})^T(A\mathbf{u}) = (\mathbf{u}^TA^T)(A\mathbf{u}) = \mathbf{u}^TA^TA\mathbf{u}$.

13. If $\mathbf{u} = k\mathbf{v}$ with $k \geq 0$ (or $\mathbf{v} = k\mathbf{u}$ with $k \geq 0$).

    Geometrically, under these conditions you can picture $\mathbf{u}$ and $\mathbf{v}$ placed head-to-tail forming one straight line, where the total length of the line equals the sum of the lengths of its two pieces.

Algebraically:

$||k\mathbf{v} + \mathbf{v}|| = ||(k+1)\mathbf{v}|| = |k+1| \; ||\mathbf{v}||$

and $||k\mathbf{v}|| + ||\mathbf{v}|| = |k| \; ||\mathbf{v}|| + ||\mathbf{v}|| = (|k|+1) \, ||\mathbf{v}||$

These two lines will be equal if k is positive.

14. Use the fact that $||A\mathbf{v}||^2 = \mathbf{v}^{\mathrm{T}} A^{\mathrm{T}} A \mathbf{v}$ (as was established in question 12f). Then:

$$\begin{aligned} ||A\mathbf{v}||^2 &= \mathbf{v}^{\mathrm{T}} A^{\mathrm{T}} A \mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}} I \mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}} \mathbf{v} \\ &= ||\mathbf{v}||^2 \end{aligned}$$

So we have $||A\mathbf{v}||^2 = ||\mathbf{v}||^2$; now take the square root of both sides.

15. You just need to write out $||A\mathbf{v}||$ and simplify it; you'll see that eventually it turns into $||\mathbf{v}||$. Alternatively, you could show that $A^{\mathrm{T}} A = I$ and use the result from question 14.

Chapter 5.2

5. First of all, you should recognize R as the rotation matrix from chapter 1.9 (page 84). So already you know what the answer will be; the angle between $\mathbf{x}$ and $R\mathbf{x}$ is θ. Now let's show this using the dot product!

$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $R\mathbf{x} = \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$

Now we use the formula for relating the dot product with the angle between two vectors:

$\mathbf{x} \cdot (R\mathbf{x}) = ||\mathbf{x}|| \, ||R\mathbf{x}|| \cos\beta \qquad$ (where β is the angle between $\mathbf{x}$ and $R\mathbf{x}$)

Working first with the left-hand side:

$$\begin{aligned} \mathbf{x} \cdot (R\mathbf{x}) &= a(a\cos\theta - b\sin\theta) + b(a\sin\theta + b\cos\theta) \\ &= a^2\cos\theta - ab\sin\theta + ab\sin\theta + b^2\cos\theta \\ &= (a^2+b^2)\cos\theta \end{aligned}$$

On the right-hand side, the $||R\mathbf{x}||$ is long to simplify so let's do that one by itself first:

$$\begin{aligned} ||R\mathbf{x}|| &= \sqrt{(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2} \\ &= \sqrt{a^2\cos^2\theta - 2ab\cos\theta\sin\theta + b^2\sin^2\theta + a^2\sin^2\theta + 2ab\cos\theta\sin\theta + b^2\cos^2\theta} \\ &= \sqrt{a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta} \\ &= \sqrt{a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

Now working with the full right-hand side:

$$\begin{aligned} ||\mathbf{x}|| \, ||R\mathbf{x}|| \cos\beta &= \left(\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}\right)\cos\beta \\ &= (a^2+b^2)\cos\beta \end{aligned}$$

Finally putting the left side and right side together again, we have:

$$\begin{aligned} (a^2+b^2)\cos\theta &= (a^2+b^2)\cos\beta \\ \cos\theta &= \cos\beta \end{aligned}$$

and since θ and β are both understood to be between 0 and 180 degrees, we have $\beta = \theta$.

6. (d) Solve $||\mathbf{v}-\mathbf{u}|| = 1$. Answer: $k = 3$

10. (a) $\pi/6$ and $5\pi/6$

    (b) Any angle between $\pi/3$ and $2\pi/3$, inclusive.

    (c) Let $\mathbf{v}$ be a unit vector. (We can do this with no loss of generality, since we're interested in the vector's direction, not its length.)

    The entries in $\mathbf{v}$ are also its direction cosines; if $\mathbf{v}$ makes angles of $\pi/6$, $\pi/6$, and θ with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively, then $\mathbf{v} = \begin{bmatrix} \cos(\pi/6) \\ \cos(\pi/6) \\ \cos\theta \end{bmatrix}$.

    Now, we said $\mathbf{v}$ was a unit vector, so $||\mathbf{v}|| = 1$, which means:

$$\begin{aligned} \sqrt{\cos^2(\pi/6) + \cos^2(\pi/6) + \cos^2\theta} &= 1 \\ 2\cos^2(\pi/6) + \cos^2\theta &= 1^2 \\ 2\left(\frac{\sqrt{3}}{2}\right)^2 + \cos^2\theta &= 1 \\ \cos^2\theta &= -\frac{1}{2} \\ \cos\theta &= \pm\sqrt{-\frac{1}{2}} \end{aligned}$$

    So there's no solution, which means that the situation as described is impossible; $\mathbf{v}$ cannot make angles of $\pi/6$ with both $\mathbf{i}$ and $\mathbf{j}$ simultaneously. (If you play with a three-dimensional model you should also be able to convince yourself of this geometrically.)

11. $$\begin{aligned} &||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 \\ &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\ &= (\mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}) \\ &= 2\mathbf{x}\cdot\mathbf{x} + 2\mathbf{y}\cdot\mathbf{y} \\ &= 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2) \end{aligned}$$

12. If $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}-\mathbf{v}||$, then

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= ||\mathbf{u}-\mathbf{v}||^2 \\ (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} &= \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ 2\mathbf{u}\cdot\mathbf{v} &= -2\mathbf{u}\cdot\mathbf{v} \\ 4\mathbf{u}\cdot\mathbf{v} &= 0 \\ \mathbf{u}\cdot\mathbf{v} &= 0 \end{aligned}$$

    so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

13. It's not true. Just about any pair of non-equal non-zero vectors in $\mathbb{R}^n$ will work for a counter-example.

14. We need to show that the angle between $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ is equal to the angle between $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$. We start by assuming that $\mathbf{x}$ and $\mathbf{y}$ have the same length, so let $||\mathbf{x}|| = ||\mathbf{y}|| = c$.

    Then the angle α between $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ can be found as follows:

$$
\begin{aligned}
\mathbf{x}\cdot(\mathbf{x}+\mathbf{y}) &= ||\mathbf{x}||\,||\mathbf{x}+\mathbf{y}||\cos\alpha && \text{(by the definition of the dot product)}\\
\mathbf{x}\cdot\mathbf{x}+\mathbf{x}\cdot\mathbf{y} &= ||\mathbf{x}||\,||\mathbf{x}+\mathbf{y}||\cos\alpha\\
||\mathbf{x}||^2+\mathbf{x}\cdot\mathbf{y} &= ||\mathbf{x}||\,||\mathbf{x}+\mathbf{y}||\cos\alpha\\
c^2+\mathbf{x}\cdot\mathbf{y} &= c\,||\mathbf{x}+\mathbf{y}||\cos\alpha && \text{(since we earlier defined the length of } \mathbf{x} \text{ to be } c)\\
\alpha &= \cos^{-1}\left(\frac{c^2+\mathbf{x}\cdot\mathbf{y}}{c\,||\mathbf{x}+\mathbf{y}||}\right)
\end{aligned}
$$

Similarly, the angle β between $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ can be found as follows:

$$
\begin{aligned}
\mathbf{y}\cdot(\mathbf{x}+\mathbf{y}) &= ||\mathbf{y}||\,||\mathbf{x}+\mathbf{y}||\cos\beta && \text{(by the definition of the dot product)}\\
\mathbf{y}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{y} &= ||\mathbf{y}||\,||\mathbf{x}+\mathbf{y}||\cos\beta\\
||\mathbf{y}||^2+\mathbf{x}\cdot\mathbf{y} &= ||\mathbf{y}||\,||\mathbf{x}+\mathbf{y}||\cos\beta\\
c^2+\mathbf{x}\cdot\mathbf{y} &= c\,||\mathbf{x}+\mathbf{y}||\cos\alpha && \text{(since we earlier defined the length of } \mathbf{y} \text{ to be } c)\\
\beta &= \cos^{-1}\left(\frac{c^2+\mathbf{x}\cdot\mathbf{y}}{c\,||\mathbf{x}+\mathbf{y}||}\right)
\end{aligned}
$$

Therefore, $\alpha=\beta$, which is what we needed to show.

15. We need to prove that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x}$.

    Let $\mathbf{x}=(x_1,x_2,...,x_n)$ and $\mathbf{y}=(y_1,y_2,...,y_n)$. Then:

    $\mathbf{x}\cdot\mathbf{y}$
    $=x_1y_1+x_2y_2+...+x_ny_n$
    $=y_1x_1+y_2x_2+...+y_nx_n$ (because regular multiplication of real numbers is commutative)
    $=\mathbf{y}\cdot\mathbf{x}$

16. a) $\mathbf{u}^T\mathbf{v}=(\mathbf{u}^T\mathbf{v})^T$ because $\mathbf{u}^T\mathbf{v}$ is a one-by-one matrix, so it doesn't change when you transpose it.

    It's a general property of the transpose of a matrix that $(\mathbf{ab})^T=\mathbf{b}^T\mathbf{a}^T$ and also that $(\mathbf{a}^T)^T=\mathbf{a}$, so in our case $(\mathbf{u}^T\mathbf{v})^T=\mathbf{v}^T(\mathbf{u}^T)^T=\mathbf{v}^T\mathbf{u}$.

    b) What's different here is that $\mathbf{u}^T\mathbf{v}$ is an $n\times n$ matrix instead of a one-by-one matrix, so it probably is not invariant under transposition. (In other words: the first equal sign is unjustified.)

Chapter 5.3

1. Note that the parametric-vector form of an equation is not unique, so it's entirely possible that your answer will look different from the one given in the book and yet still be correct. Just check that your direction vector is a scalar multiple of the one in the given answer, and verify that your $\mathbf{x}_0$ does in fact lie on the line.

3. c. Any point on the line $\mathcal{L}$ is of the form $\mathbf{x}=\begin{bmatrix}-3+t\\1-2t\end{bmatrix}$. The distance between points $\mathbf{a}$ and $\mathbf{b}$ is $||\mathbf{a}-\mathbf{b}||$, so we want $\left|\left|\begin{bmatrix}-3+t\\1-2t\end{bmatrix}-\begin{bmatrix}-3\\1\end{bmatrix}\right|\right|$ to equal 2.

   Thus $\left|\left|\begin{bmatrix}t\\-2t\end{bmatrix}\right|\right|=2$, giving $\sqrt{t^2+4t^2}=2$, so finally $t=\pm\dfrac{2}{\sqrt{5}}$.

5. First, the two direction vectors need to be scalar multiples of each other, so $b = 8$.

   Next, note that if the two equations represent the same line, then
   $\begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ must be consistent, so $a = -1$.

10. Remember that $\mathbf{k} = (0, 0, 1)$. If we construct a vector $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{k} = 0$, then $\mathbf{x} = t\mathbf{v}$ will be a line with the desired property. For $\mathbf{v}$, you can use $(a, b, 0)$ with a and b being any real numbers.

12. (a) The line in $\mathbb{R}^n$ which contains $\mathbf{p}$ and $\mathbf{q}$ is:

$$\begin{aligned} \mathbf{x} &= \mathbf{p} + t(\mathbf{q} - \mathbf{p}) \quad && \text{(because } \mathbf{p} \text{ is a point on the line and} \\ & && \mathbf{q} - \mathbf{p} \text{ is a vector parallel to the line)} \\ &= \mathbf{p} + t\mathbf{q} - t\mathbf{p} \\ &= \mathbf{p} - t\mathbf{p} + t\mathbf{q} \\ &= (1 - t)\mathbf{p} + t\mathbf{q} \end{aligned}$$

    (b) $8x + y = 11$

13. This question was accidentally omitted from the Vector Geometry notes when they were included in the textbook. I think it's a nice question, so I'll include it here. The question is: Let $\{\mathbf{p}, \mathbf{q}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Describe the difference between the following expressions (where the parameters s and t are assumed to range over all real numbers).

    (a) $s\mathbf{p} + t\mathbf{q}$
    (b) $s\mathbf{p} + (1 - s)\mathbf{q}$
    (c) $s\mathbf{p} + \mathbf{q}$

    And the answers are:

    (a) It's a plane through the origin.
    (b) It's a line through $\mathbf{p}$ and $\mathbf{q}$.
    (c) It's a line through $\mathbf{q}$, parallel to $\mathbf{p}$.

Chapter 5.4

4. A basis for the subspace is: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   An equation for the line through the origin normal to the hyperplane is $\mathbf{x} = t \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}$.

7. (b) $\theta = \cos^{-1}(2/3)$

8. (a) $x + y = 2$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   (b) Answer is in the back of the Vector Geometry notes

   (c) $x_1 + x_2 + x_3 + x_4 = 4$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   (d) $x_1 + x_2 + x_3 + x_4 + x_5 = 5$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

9. (a) Answer is in the back of the Vector Geometry notes

   (b) They don't intersect.

   (c) (6.5, 4.5, 1.5)

   (d) Answer is in the back of the Vector Geometry notes

   (e) (2, 0, -1, -1)

11. All you have to do is solve the appropriate systems of equations and write the solution sets in parametric vector form. So for example, for part (a), you'd solve the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 2 \end{aligned}$$

13. It's a parametric vector equation in two free variables (s and t), so it does represent a plane.

    When $s = 0$ and $t = 0$, $\mathbf{x} = \mathbf{p}$, so the plane contains $\mathbf{p}$.

    When $s = 1$ and $t = 0$, $\mathbf{x} = \mathbf{q}$, so the plane contains $\mathbf{q}$.

    When $s = 0$ and $t = 1$, $\mathbf{x} = \mathbf{r}$, so the plane contains $\mathbf{r}$.

14. (a) The normal form is easier; just sub in the values and see if it works.

    (b) The parametric vector form is easier; just arbitrarily pick 5 different values for s and t, and use them to generate points.

Chapter 5.6

8. Start with the left side and manipulate it until it turns into the right side:

$$\begin{aligned} & (\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) & \\ = \; & (\mathbf{u}+\mathbf{v}) \times \mathbf{u} + (\mathbf{u}+\mathbf{v}) \times (-\mathbf{v}) & \text{(distributive property)} \\ = \; & \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} + \mathbf{u} \times (-\mathbf{v}) + \mathbf{v} \times (-\mathbf{v}) & \text{(distributive property again)} \\ = \; & \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{v} & \text{(because } \mathbf{a} \times \mathbf{b} = (-\mathbf{b}) \times \mathbf{a}) \\ = \; & \mathbf{0} + \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} + \mathbf{0} & \text{(because } \mathbf{a} \times \mathbf{a} = \mathbf{0}) \\ = \; & 2\mathbf{v} \times \mathbf{u} & \end{aligned}$$

9. There is a proof of this on page 205 of your textbook (*Linear Algebra* by David C. Lay), with the vectors forming the columns of the matrix rather than the rows – then just recall that the determinant doesn't change when you transpose a matrix.

   Here's another proof, using methods from this chapter (you'll notice the resemblance to the proof I did in class to show that $||\mathbf{u} \times \mathbf{v}||$ = the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$):

   We know, from trigonometry, that the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ is $||\mathbf{u}||\,||\mathbf{v}||\sin\theta$. (For illustration, see Figure 5.11 from page 324 of your textbook, or your notes from class.)

   So then the area *squared* equals

$$
\begin{aligned}
& ||\mathbf{u}||^2\,||\mathbf{v}||^2 \sin^2\theta \\
= \;& ||\mathbf{u}||^2\,||\mathbf{v}||^2 (1-\cos^2\theta) \\
= \;& ||\mathbf{u}||^2\,||\mathbf{v}||^2 - ||\mathbf{u}||^2\,||\mathbf{v}||^2 \cos^2\theta \\
= \;& ||\mathbf{u}||^2\,||\mathbf{v}||^2 - (||\mathbf{u}||\,||\mathbf{v}||\cos\theta)^2 \\
= \;& ||\mathbf{u}||^2\,||\mathbf{v}||^2 - (\mathbf{u}\cdot\mathbf{v})^2 \\
= \;& (u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 \\
= \;& u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2 - u_1^2v_1^2 - 2u_1u_2v_1v_2 - u_2^2v_2^2 \\
= \;& (u_1v_2)^2 - 2u_1u_2v_1v_2 + (u_2v_1)^2 \\
= \;& (u_1v_2 - u_2v_1)^2 \qquad\qquad \text{(by factoring)}
\end{aligned}
$$

   So now we have $(\text{Area})^2 = (u_1v_2 - u_2v_1)^2$. Note that the right hand side is exactly the square of the determinant of the 2×2 matrix A that has $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ for its rows. Then taking the square root of both sides and remembering that the determinant might not be positive, we have proved that the area equals $|\det A|$.

11. Use the fact that the volume of the paralellepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ equals the absolute value of the scalar triple product, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

    So the volume of the second paralellepiped will be equal to the absolute value of the scalar triple product $\mathbf{u}\cdot((\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u}))$. Then:

$$
\begin{aligned}
& \mathbf{u}\cdot((\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u})) \\
= \;& \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w} - \mathbf{u}\times\mathbf{w} - \mathbf{v}\times\mathbf{u} + \mathbf{u}\times\mathbf{u}) && \text{(using distributive property twice)} \\
= \;& \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) - \mathbf{u}\cdot(\mathbf{u}\times\mathbf{w}) - \mathbf{u}\cdot(\mathbf{v}\times\mathbf{u}) + \mathbf{u}\cdot(\mathbf{u}\times\mathbf{u}) && \text{(distributive prop. of dot product)}
\end{aligned}
$$

    The three last terms all become zero (being the dot products of orthogonal vectors), so we're left with just $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

13. First of all, notice that A is the matrix (mentioned in chapter 5.6) with the property that if $\mathbf{u} = (a, b, c)$ then $\mathbf{u}\times\mathbf{x} = A\mathbf{x}$.

    Now let's find $\operatorname{Nul} A$. Start by assuming that $c \neq 0$. Then we row-reduce:

$$
\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \sim \begin{bmatrix} c & 0 & -a \\ 0 & -c & b \\ -bc & ac & 0 \end{bmatrix} \sim \begin{bmatrix} c & 0 & -a \\ 0 & -c & b \\ 0 & ac & -ab \end{bmatrix} \sim \begin{bmatrix} c & 0 & -a \\ 0 & -c & b \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & -a/c \\ 0 & 1 & -b/c \\ 0 & 0 & 0 \end{bmatrix} \text{ so } \operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} a/c \\ b/c \\ 1 \end{bmatrix} \right\} \text{ or, equivalently, Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}.
$$

In other words, $\mathbf{u} \times \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{u}$.

But remember we assumed above that $c \neq 0$. Now let's assume that $c = 0$ but $b \neq 0$ (so $\mathbf{u} = (a, b, 0)$). Then

$$A = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & -a \\ -b & a & 0 \end{bmatrix} \sim \begin{bmatrix} -b & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & b \end{bmatrix} \sim \begin{bmatrix} -b & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -a/b & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\operatorname{Nul} A = \operatorname{Span}\left\{\begin{bmatrix} a/b \\ 1 \\ 0 \end{bmatrix}\right\}$ or, equivalently, $\operatorname{Span}\left\{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}\right\}$. Again, $\mathbf{u} \times \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{u}$.

Next, if $c = 0$ and $b = 0$ but $a \neq 0$ (so $\mathbf{u} = (a, 0, 0)$) we have:

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $\operatorname{Nul} A = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ or $\operatorname{Span}\left\{\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}\right\}$,
and again we can say that $\mathbf{u} \times \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{u}$.

The only remaining possibility is that $\mathbf{u} = \mathbf{0}$, in which case A is the zero matrix, $\operatorname{Nul} A$ is all of $\mathbb{R}^3$, and $\mathbf{u} \times \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$. (In this case $\mathbf{x}$ may not be a scalar multiple of $\mathbf{u}$, however $\mathbf{u}$ is certainly a scalar multiple of $\mathbf{x}$, that is $0\mathbf{x}$.)

So the property of the cross product illustrated by calculating $\operatorname{Nul} A$ is: $\mathbf{u} \times \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{x}$ are parallel.

14. METHOD 1: Start by observing that:

$$\begin{aligned} & (\mathbf{u}+\mathbf{v}+\mathbf{w}) \times \mathbf{u} \\ = {} & (\mathbf{u} \times \mathbf{u}) + (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u}) && \text{(distributive property)} \\ = {} & \mathbf{0} - (\mathbf{u} \times \mathbf{v}) + (\mathbf{w} \times \mathbf{u}) && \text{(properties of the cross product)} \\ = {} & \mathbf{0} - (\mathbf{w} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u}) && \text{(by hypothesis)} \\ = {} & \mathbf{0} \end{aligned}$$

Having shown that $(\mathbf{u}+\mathbf{v}+\mathbf{w}) \times \mathbf{u} = \mathbf{0}$, we can conclude that $(\mathbf{u}+\mathbf{v}+\mathbf{w})$ and $\mathbf{u}$ are parallel.

It was given that $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, so we know that $\mathbf{u} \neq \mathbf{0}$, so we can say that $(\mathbf{u}+\mathbf{v}+\mathbf{w}) = k_1\mathbf{u}$ for some scalar k_1.

Similarly, we can show that $(\mathbf{u}+\mathbf{v}+\mathbf{w}) \times \mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, so we can say that $(\mathbf{u}+\mathbf{v}+\mathbf{w}) = k_2\mathbf{v}$ for some scalar k_2.

So now we have $k_1\mathbf{u} = k_2\mathbf{v}$. But $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, which means that $\mathbf{u}$ and $\mathbf{v}$ cannot be parallel; therefore k_1 and k_2 must both be 0, which means that $(\mathbf{u}+\mathbf{v}+\mathbf{w}) = 0\mathbf{u} = \mathbf{0}$.

METHOD 2: Since $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$, we can conclude that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ all lie in the same plane.

Since $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, we know that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, so we can write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$:

$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$

Then taking the cross product with $\mathbf{u}$ on both sides:

$\mathbf{u} \times \mathbf{w} = \mathbf{u} \times (c_1\mathbf{u} + c_2\mathbf{v})$

$\mathbf{u} \times \mathbf{w} = c_1(\mathbf{u} \times \mathbf{u}) + c_2(\mathbf{u} \times \mathbf{v})$

$\mathbf{u} \times \mathbf{w} = \mathbf{0} + c_2(\mathbf{u} \times \mathbf{v})$

$-\mathbf{w} \times \mathbf{u} = \mathbf{0} + c_2(\mathbf{u} \times \mathbf{v})$

and then, since $\mathbf{w} \times \mathbf{u} = \mathbf{u} \times \mathbf{v}$ by hypothesis, we have:

$-\mathbf{u} \times \mathbf{v} = \mathbf{0} + c_2(\mathbf{u} \times \mathbf{v})$

so $c_2 = -1$.

Similarly, we can show that $c_1 = -1$, so $\mathbf{w} = -\mathbf{u} - \mathbf{v}$, so

$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{u} + \mathbf{v} - \mathbf{u} - \mathbf{v} = \mathbf{0}$.